# The free energy of hot QCD

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*Based on K. Kajantie, M. Laine, K. Rummukainen, Y. Schröder, hep-ph/0007109, to appear in Phys. Rev. Lett.*

Consider static thermodynamic observables in QCD.

Free energy: $F(T,V) = -T \ln \mathrm{Tr}\, e^{-\beta H}, \quad \beta = T^{-1}$.

Pressure: $p(T) = -\frac{\partial F(T,V)}{\partial V} = -\frac{F}{V}$.

Entropy density: $s(T) = \frac{dp}{dT}$.

Energy density: $e(T) = Ts(T) - p(T)$.

Spatial correlation lengths: $\mathrm{Tr}\, O(\mathbf{x})O(0)e^{-\beta H} \propto e^{-\frac{|\mathbf{x}|}{\xi(T)}}$.

For all such quantities one can write down a Euclidian path integral expression. For instance,

$$
\begin{aligned}
F(T,V) &= -T \ln \int_{\text{b.c.}} \mathcal{D}[A_\mu^a, \bar\psi, \psi] e^{-\int_0^\beta d\tau \int d^3x \mathcal{L}[A_\mu^a, \bar\psi, \psi]}, \\
\mathcal{L} &= \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} + \bar\psi \gamma_\mu D_\mu \psi,
\end{aligned}
$$

and b.c. over $\tau$ are periodic ($A_\mu^a$) or anti-periodic ($\bar\psi, \psi$).

Why are these quantities of interest?

Recall for instance the cosmological Big Bang context. From the Einstein equations it follows that

$$\frac{dT}{dt} = -\frac{\sqrt{24\pi}}{m_{\,\mathsf{Pl}}}\frac{\sqrt{e(T)}}{d\ln s(T)/dT}.$$

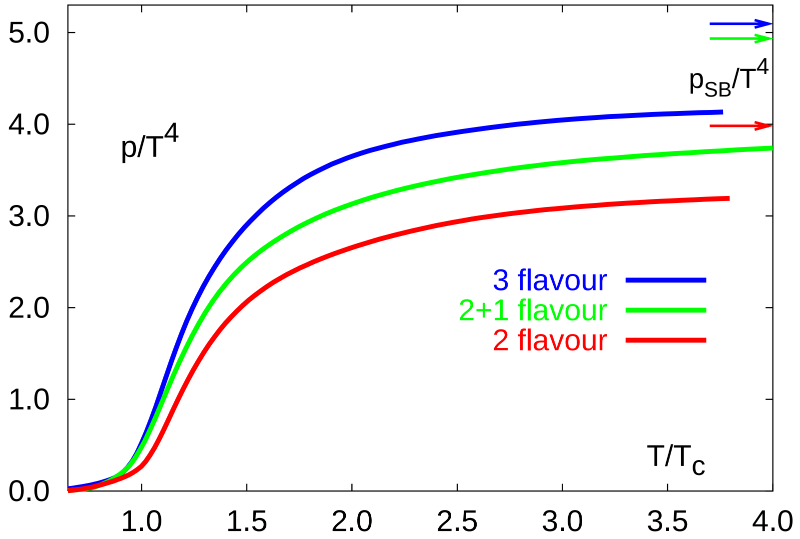
The essential structure here is given by QCD below the electroweak scale $T \sim 100$ GeV, $t \sim 10^{-12}$ s.

In addition, a singularity at $T_c \sim 200$ MeV could result in a definite spectrum of macroscopic fluctuations (at a current scale of about 1 ly), which could affect nucleosynthesis, primordial magnetic fields, structure formation.

Hopefully $p(T)$, $e(T)$, $s(T)$ are also experimentally testable in Little Bang of heavy ion collisions, for some temperature range.

How well do we know $p(T)$ at the moment?

Lattice simulation results from Karsch et al, hep-lat/0010027:




Open problems:

- Normalization at $T < T_c$?
  (Only $p(T) - p(T_0)$ can be measured.)
- Value of $T_c/\Lambda_{\overline{\mathrm{MS}}}$?
  (Conversion to physical units and continuum limit.)
- Is there a singularity at $T_c$ in the physical case?
  (Thermodynamic limit.)

The last two questions can be answered with more simulations.

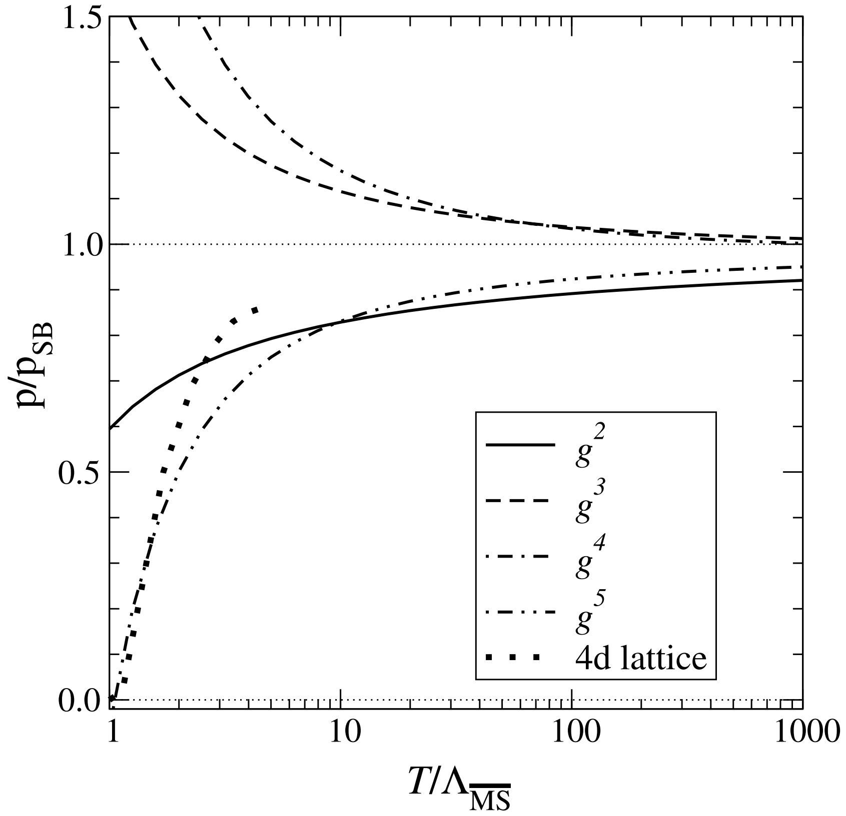
- What happens at $T \gtrsim 4T_c$? How does one approach the ideal Stefan-Boltzmann limit?

Perturbation theory in $g$ is not a practical guide either!

$$\frac{p}{p_{\mathsf{SB}}} = 1 + c_1 g^2 + c_2 g^3 + c_3 g^4 \ln\frac{1}{g} + c_4 g^4 + c_5 g^5 + c_6 g^6 \ln\frac{1}{g} + c_7 g^6 + \ldots$$

$c_1$: Shuryak '78 (2-loop)
$c_2$: Kapusta '79 (resummed 2-loop)
$c_3$: Toimela '83 (resummed 2-loop)
$c_4$: Arnold, Zhai '94 (resummed 3-loop)
$c_5$: Kastening, Zhai '95 (resummed 3-loop)
$c_6$: not computed yet (4-loop)
$c_7$: not computable even in principle! (confinement...)

$N_f = 0$:



How can one understand the breakdown? (even for $g \ll 1$)

Question analysed by Braaten and Nieto, hep-ph/9510408.

The answer: it is a multiscale system. There are thermal fluctuations at the momentum scale $p \sim 2\pi T$, but also soft collective modes at $p \sim gT, g^2T$.

Expansion parameters related to different scales are different:

$$\epsilon \sim \frac{1}{\pi} g^2(2\pi T) n_b(p) = \frac{g^2(2\pi T)}{\pi(e^{p/T} - 1)} \overset{p<T}{\sim} \frac{g^2(2\pi T)T}{\pi p}.$$

Then $p \sim 2\pi T$ is still OK, but the other expansion parameters are large, and give odd powers, or no powers at all.

The other way to say it is that the soft momentum scales $p \sim gT, g^2T$ are formally described by a 3d SU($N_c$) gauge theory (see below), which is *confining*.

In order to improve on the convergence we need a reorganisation of the perturbative series, *a resummation*.

Andersen, Braaten, Strickland;
Blaizot, Iancu, Rebhan;
Peshier; Parwani; ...

However difficult to carry out systematically to high orders, or if genuinely non-perturbative ($\sim$"confining") effects arise.

A systematic way is to construct a set of *effective theories*. Then only need to deal with convergent integrals at any step.

Original theory:
$|\mathbf{p}| \sim g^2T, gT, 2\pi T$, $\omega \sim g^4T, gT, 2\pi T$.

Hard Thermal Loop effective theory: Pisarski, Braaten, ...
$|\mathbf{p}| \sim g^2T, gT$, $\omega \sim g^4T, gT$.

Classical Boltzmann/Langevin equations: Bödeker; Arnold, Son, Yaffe; Blaizot, Iancu; ...
$|\mathbf{p}| \sim g^2T$, $\omega \sim g^4T$.

Dimensional Reduction: Ginsparg; Appelquist, Pisarski; ...
$|\mathbf{p}| \sim g^2T, gT$, $\omega \equiv 0$.

Or, if $T \gg 10^3 T_c$: Ginsparg; Appelquist, Pisarski; ...
$|\mathbf{p}| \sim g^2T$, $\omega \equiv 0$.

However, $g$ must be small! The effective coupling turns out to be $\sim g^2(2\pi T)$, but how low can one go in practice?

The idea of dimensional reduction: write the compact time direction in terms of discrete Fourier (Matsubara) modes.

Fermions: $\omega_n = i\pi T(2n+1)$, $n \in \mathbb{Z}$.
Bosons: $\omega_n = i2\pi T n$, $n \in \mathbb{Z}$.

Looking at the structure of the propagators,

$$\frac{1}{-\omega_n^2 + |\mathbf{p}|^2},$$

we note that modes with $\omega_n \neq 0$ are effectively massive particles living in 3 Euclidian dimensions. They can be integrated out.

Write then down the most general gauge-invariant effective Lagrangian for the bosonic zero modes, $\omega_n = 0$:

$$\mathcal{L}_{\text{3d}} = \frac{1}{2}\operatorname{Tr} F_{ij}^2 + \operatorname{Tr}[D_i, A_0]^2 + m_D^2 \operatorname{Tr} A_0^2 + \lambda_A (\operatorname{Tr} A_0^2)^2.$$

Here $m_D^2 \sim g^2T^2$, $\lambda_A \sim g^4 T$ are generated radiatively.
We do not want to include higher order operators!

One of the strengths of specialising to $\omega \equiv 0$ is that it puts the theory directly in a simple form where decoupling conjectures familiar from zero temperature field theory can be applied.

Indeed, an explicit computation shows that the lowest-dimensional higher order operators are of the form

$$\mathcal{O} \sim \frac{g_3^n}{T^{\mathbf{3}}}(A_i, A_0)^n |\mathbf{p}|^{6-n},$$

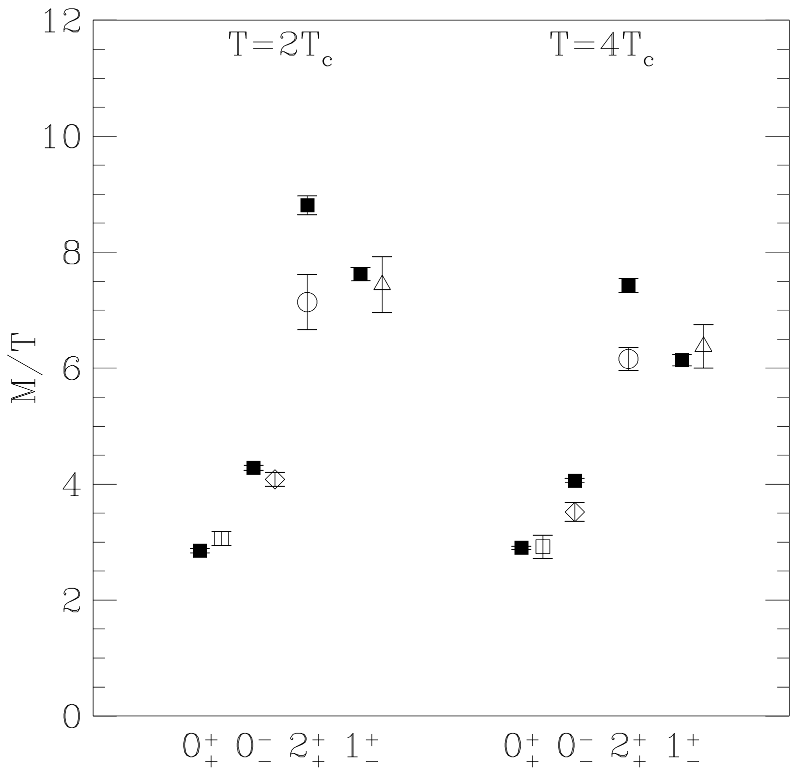
where $n \geq 2$. (E.g., $\mathcal{O} \sim (g_3^2/\pi^4 T^{\mathbf{3}})\,\mathrm{Tr}[D_i, F_{ij}]^2$).

When neglected, Symanzik's conjecture says that the relative error made in a *generic Green's function* is $(|\mathbf{p}|/T)^{\mathbf{3}} \lesssim g^{\mathbf{3}}$.

This *universality* should be stressed. The theory applies to an infinite set of observables. Thus the effective theory can really be said to provide for a physical description of the system.

This was all parametric, and the physical coupling is not that small. So test the accuracy for SU(2) where 4d lattice measurements are most precise: compare correlation lengths!

Let $M = \xi^{-1}$. Measured by Hart, Philipsen, hep-lat/9908041.




(Open symbols are 4d finite $T$, closed 3d effective theory.)

Observations:

1. Comparison tests directly the value of $g_3^2/T$. Accuracy seems to be good even down to $T \sim 2T_c$! Scale dependence of NLO $g_3^2/T$ with $\mu = (0.5...2.0)\mu_{\text{opt}}$ would give 7%.

2. Problems closer to $T_c$. For $N_f = 0$, Z($N_c$) symmetry driving the transition is not reproduced by the effective theory (it is related to momenta $\sim 2\pi T$). For $N_f > 0$, chiral symmetry driving the transition is not reproduced by the effective theory (mesonic states have momenta $\sim 2\pi T$).

3. For $T \gtrsim 2T_c$ we may expect the accuracy of DR to be *better* than in the effective theories for $\omega > 0$, since corrections to the action and parameters have been included systematically.

4. Correlation lengths are completely non-perturbative, unless $T \gg 10^7 T_c$! For instance,

$$m_D^{\text{phys}} = (\frac{N_c}{3}+\frac{N_f}{6})^{\frac{1}{2}} gT + \frac{N_c}{4\pi} g^2 T [\ln \frac{(N_c/3+N_f/6)^{\frac{1}{2}}}{g} + 7.0] + \mathcal{O}(g^3 T),$$

where 7.0 is a non-pert. constant, determined with simulations.

Apply this setup now to the QCD pressure at $T \gtrsim 2T_c$. Computing at NLO and using "fastest apparent convergence" criterion for higher order terms, we get (for $N_f = 0, N_c = 3$)

$$
\begin{aligned}
\frac{g_3^2}{T} &= \frac{8\pi^2}{11\ln(6.742T/\Lambda_{\overline{\rm MS}})}, \\
x &= \frac{\lambda_A}{g_3^2} = \frac{3}{11\ln(5.371T/\Lambda_{\overline{\rm MS}})} \sim \frac{3}{2}\frac{\alpha_s}{\pi}, \\
y &= \frac{m_D^2}{g_3^4} = \frac{3}{8\pi^2 x} + \frac{9}{16\pi^2}.
\end{aligned}
$$

Let us denote

$$
\mathcal{F}_{\overline{\rm MS}}(x,y) = -\frac{1}{Vg_3^6}\ln\int\mathcal{D}A_\mu^a e^{-\int d^3x\mathcal{L}_{\rm 3d}}.
$$

Then, up to hard mode $\mathcal{O}(g^6T^4)$ corrections, (Braaten, Nieto)

$$
\frac{p(T)}{p_{\rm SB}} = \underbrace{1 - \frac{5}{2}x}_{\mathcal{O}(g^2)} - \frac{45}{8\pi^2}\left(\frac{g_3^2}{T}\right)^3\left(\mathcal{F}_{\overline{\rm MS}}(x,y) - \underbrace{24\frac{y}{(4\pi)^2}\ln\frac{\bar\mu_{\rm 3d}}{T}}_{\mathcal{O}(g^4)}\right).
$$

The $\bar\mu_{\rm 3d}$-dependence is cancelled by that in $\mathcal{F}_{\overline{\rm MS}}(x,y)$.

However absolute value of $\mathcal{F}_{\overline{\mathrm{MS}}}(x,y)$ is not measurable, just like $F(T,V)$ is not in 4d. We need a difference:

$$\mathcal{F}_{\overline{\mathrm{MS}}}(x,y) \quad = \quad \mathcal{F}_{\overline{\mathrm{MS}}}(x_0,y_0) + \int_{y_0}^{y} dy\,(\frac{\partial \mathcal{F}_{\overline{\mathrm{MS}}}}{\partial y} + \frac{dx}{dy}\frac{\partial \mathcal{F}_{\overline{\mathrm{MS}}}}{\partial x}).$$

The partial derivatives are given by condensates:

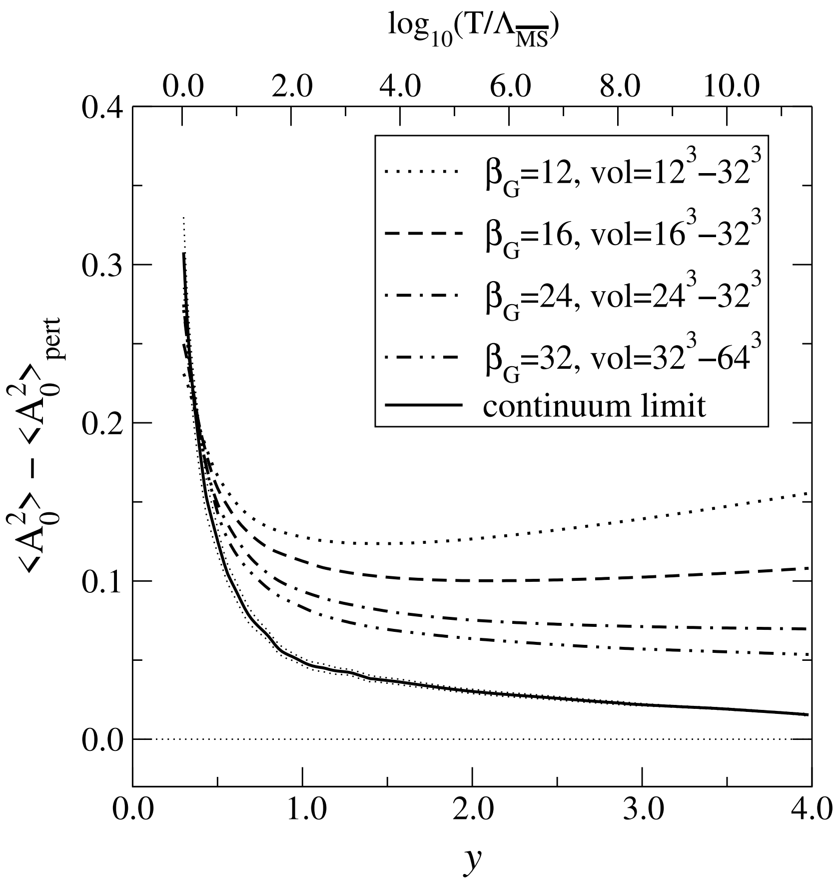
$$\frac{\partial \mathcal{F}_{\overline{\mathrm{MS}}}}{\partial y} = \langle \frac{\mathrm{Tr}\, A_0^2}{g_3^2} \rangle_{\overline{\mathrm{MS}}}.$$

In the case of $\partial \mathcal{F}_{\overline{\mathrm{MS}}}/\partial x$, a similar relation is obtained but with the condensate $\langle (\mathrm{Tr}\, A_0^2)^2 \rangle$. These are measurable, and sum up an infinite set of higher order corrections!

It turns out that the latter condensate gives parametrically and numerically a small effect (for $T \gg T_c$), so neglect it for now.

On the other hand, $\mathcal{L}_{\rm 3d}$ is super-renormalizable. With an exact (finite order) computation in lattice perturbation theory, a condensate measured in lattice Monte Carlo simulations can be related to the condensate $\langle \mathrm{Tr}\, A_0^2 \rangle_{\overline{\rm MS}}$.

$\beta_G = \frac{6}{g_3^2 a}$:



It remains to determine the integration constant $\mathcal{F}_{\overline{\rm MS}}(x_0, y_0)$. Do this in the opposite direction from 4d simulations!

The idea is that at high enough $T$, $g(2\pi T) \gg g^2(2\pi T)$, and the effective theory can be simplified further by integrating out $A_0$. Then there remains a contribution of the simple form

$$\Delta\mathcal{F}_{\overline{\rm MS}} = -\frac{1}{V g_3^6} \ln \int \mathcal{D}A_i^a e^{-\int d^3x \frac{1}{2} \operatorname{Tr} F_{ij}^2}.$$

This is the famous non-perturbative $\mathcal{O}(g^6 T^4)$ Linde term.

But remarkably this absolute value *can be* measured, since the theory only contains one parameter $g_3^2$, and it is dimensionful! $\Rightarrow$ (renormalized) $\Delta\mathcal{F}_{\overline{\rm MS}}$ is fully determined by $\langle \frac{1}{2} \operatorname{Tr} F_{ij}^2 \rangle$.

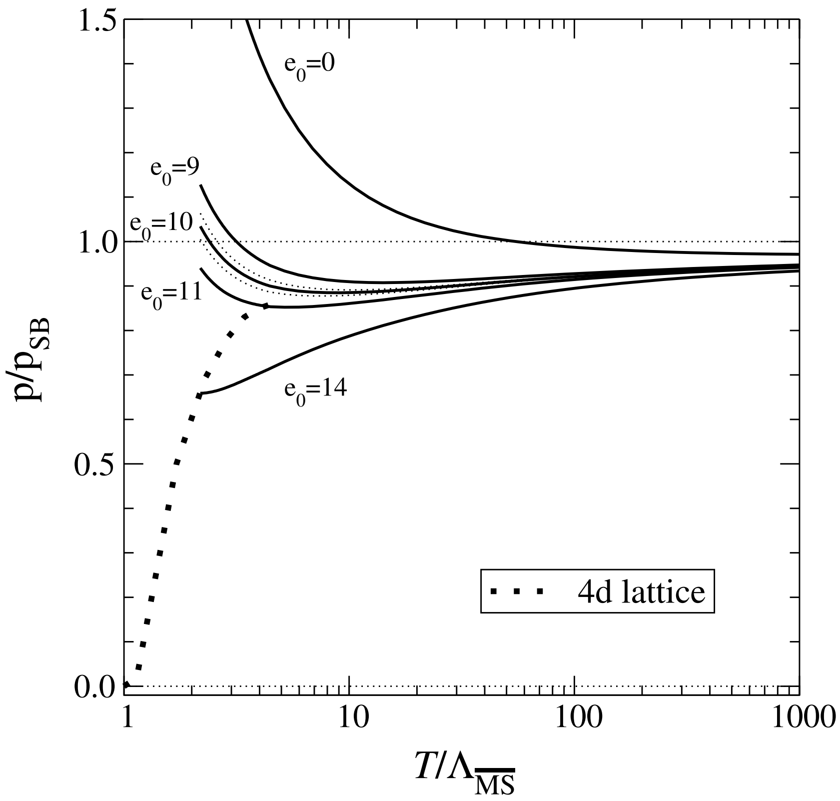
**So in principle we are done!**

In practice, though, we have to parameterise the single number $\Delta\mathcal{F}_{\overline{\rm MS}}$, since the lattice-continuum relation here is of 4-loop order and has not been worked out yet:

$$\Delta\mathcal{F}_{\overline{\rm MS}} = \frac{e_0}{(4\pi)^4} d_A C_A^3,$$

where $d_A = N_c^2 - 1, C_A = N_c$. (Same problem with $\langle (\operatorname{Tr} A_0^2)^2 \rangle$)

Current status (only $\langle \mathrm{Tr}\, A_0^2 \rangle$ included):




Results:

1. Outcome depends strongly on the non-perturbative contribution of order $\mathcal{O}(g^6T^4)$, as can be observed from the $e_0$-dependence. Order of magnitude $\mathcal{O}(10)$ seems reasonable, since it is known from $m_D$ that non-perturbative constants tend to be large. It can eventually be measured directly.

2. When the $\mathcal{O}(g^6T^4)$ term is summed together with the set of all higher order terms determined via $\langle \mathrm{Tr}\, A_0^2 \rangle$, these long-distance contributions almost cancel at $T \gtrsim 30\Lambda_{\overline{\mathrm{MS}}}$!

## Extensions

Measure $\langle (\mathrm{Tr}\, A_0^2)^2 \rangle$, $\langle \mathrm{Tr}\, F_{ij}^2 \rangle$, and convert result to $\overline{\mathrm{MS}}$ (4-loop relation needed).

Inspect effect of $N_f, m_s, \ldots$

Include $\mu \neq 0$: $\Omega(T, \mu, V) = -T \ln \mathrm{Tr}\, e^{-\beta(H - \mu B)}$, where $B$ is the baryon number. The effective theory gets then a new operator,

$$\delta\mathcal{L}_{\,\text{3d}} \propto i g_3^3 N_f (\mu/T)\, \mathrm{Tr}\, A_0^3.$$

Etc.

## Conclusions

The use of a dimensionally reduced effective theory and 3d lattice simulations provides for a systematic tool of determining unambiguously all static observables, such as the free energy and correlation lengths, at $T \in (\sim 2T_c, \infty)$.

While the setup is clear, current results are not complete. We need, in particular, a number of well-defined but technically challenging high order perturbative computations.

For the free energy, we find a remarkable cancellation of infrared sensitive effects, and the final outcome is dominated by ultraviolet. Not so for many other observables, such as correlation lengths. Confinement leaves a trace even at $T > T_c$.