The pressure of hot/dense QCD

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- Motivation
- Challenges
- Current status

Consider QCD with N_c colours and N_f massless flavours:

$$\mathcal{L}_{\text{QCD}} = \frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i \gamma_\mu D_\mu \psi_i \ .$$

A baryon number B is defined as

$$B = \frac{1}{3}\int \mathrm{d}^3x \ \sum_{i=1}^{N_f} \bar{\psi}_i \gamma_0 \psi_i \ .$$

Want to compute a basic characteristic of the thermodynamics of this system, minus its free energy density, or pressure,

$$p(T,\mu_B) \equiv \lim_{V\to\infty} \frac{T}{V} \ln \mathrm{Tr} \exp[-\frac{1}{T}(\mathcal{H}_{\text{QCD}} - \mu_B B)] \ .$$

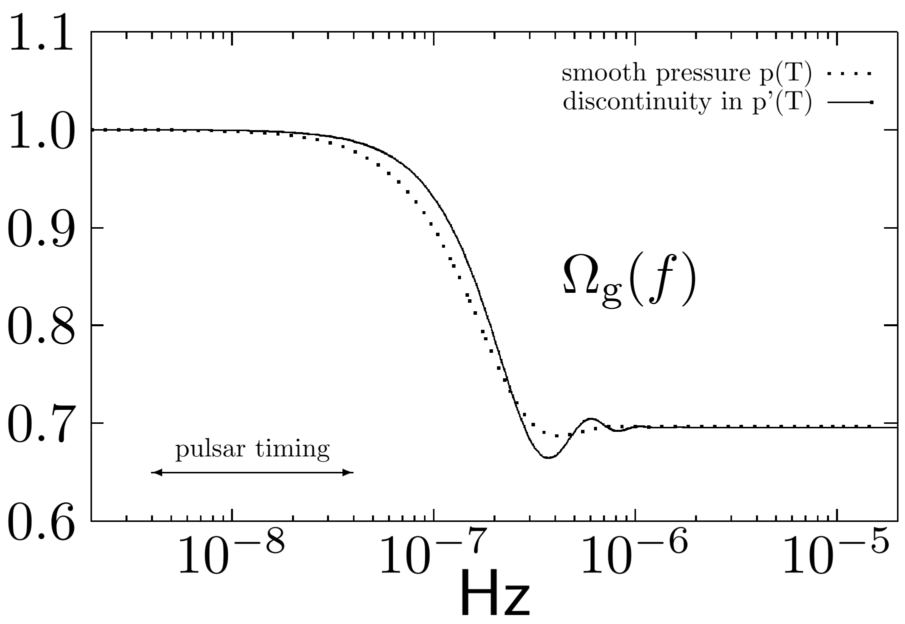
Where does $p(T, \mu_B)$ appear in physics?

In cosmology, $\mu_B/T \sim n_B/n_\gamma \sim 10^{-10}$, and $p(T) \equiv p(T, 0)$. The cooling rate of the (flat) Universe is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\sqrt{24\pi}}{m_{\text{Pl}}} \frac{\sqrt{e(T)}}{\mathrm{d}\ln s(T)/\mathrm{d}T},$$

where $s(T) = \mathrm{d}p(T)/\mathrm{d}T$, $e(T) = Ts(T) - p(T)$.

A non-trivial shape of $p(T)$ causes deviations from ideal gas behaviour. In principle this leaves a trace e.g. in the frequency spectrum of primordial gravitational waves:



D.J. Schwarz 1997

A singularity in $p(T)$ could lead to other consequences as well.

In heavy ion collision experiments, one of the most interesting theoretical and experimental issues is how fast an initially non-equilibrium system thermalises.

[This should be of interest also to inflationary cosmology.]

One way to address this is to **assume** rapid thermalisation, and see what the consequence, ideal hydrodynamics, leads to:

$$T^{\mu\nu} = [p(T) + e(T)]u^\mu u^\nu - p(T)g^{\mu\nu}, \quad \partial_\mu T^{\mu\nu} = 0 \ ,$$

where $u^\mu(t, \mathbf{x})$ is the local flow velocity, and $T = T(t, \mathbf{x})$. After hydrodynamic expansion system is hadronised at $T \sim 150$ MeV.

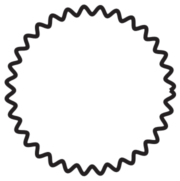
The claim is that the azimuthal anisotropy ("elliptic flow") observed in charged particle momentum distributions from Au $+$ Au collisions, supports the hypothesis of fast initial thermalisation and subsequent hydrodynamic expansion, with $\mu_B \lesssim T$ and a $p(T, \mu_B)$ with the gross features predicted by QCD.

Kolb, Sollfrank, Heinz;
Huovinen, Ruuskanen, ...

So take $T \gg$ GeV, $\mu_B = 0$, and try perturbative expansion.

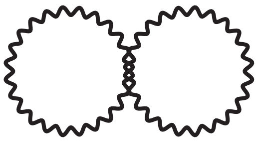
$$p(T) = \lim_{V\to\infty} \frac{T}{V} \ln \int_{\text{b.c.}} \mathcal{D}[A_\mu^a, \bar\psi, \psi] \exp\left(-\int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_{\text{QCD}}\right) .$$

1-loop: Stefan, Boltzmann



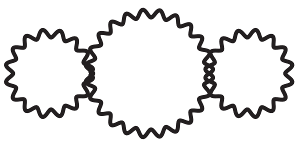
$$p(T) = \frac{\pi^2 T^4}{45}\left(N_c^2 - 1 + \frac{7}{4} N_c N_f\right) .$$

2-loop: Shuryak 1978; Chin 1978

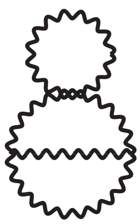


$$\ldots \qquad \delta p(T) = -\frac{g^2 T^4}{144}(N_c^2 - 1)\left(N_c + \frac{5}{4} N_f\right) .$$

3-loop: There are uncancelled infrared (IR) divergences!



$$g^4 T^5 \int \frac{\mathrm{d}^3 p}{p^4} + \ldots$$ power divergent .



$$g^4 T^4 \int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 q}{p^2 q^2 (p+q)^2} + \ldots$$ log divergent .

Thus g^2 is not a good expansion parameter even at $T \gg$ GeV!

The divergences can be cured by an identification and resummation of the most IR-sensitive terms at all orders.

The simplest way to implement the resummation is by effective field theory methods. In the present context that means high temperature dimensional reduction: integrating out the non-zero bosonic and fermionic Matsubara frequencies,

$$\omega_n^b = 2\pi T n, \quad \omega_n^f = 2\pi T(n + \frac{1}{2}) \; ,$$

yields an effective theory for the bosonic Matsubara zero modes.

Ginsparg; Appelquist, Pisarski;
Nadkarni; Kajantie et al; Braaten and Nieto;...

Effective action ("Electrostatic QCD") is the most general respecting discrete symmetries and gauge invariance under static transformations:

$$
\begin{aligned}
\mathcal{L}_{\text{E}} &= \frac{1}{2}\mathrm{Tr}\,F_{ij}F_{ij} + \mathrm{Tr}[D_i, A_0]^2 + m_{\text{E}}^2\,\mathrm{Tr}A_0^2 + \lambda_{\text{E}}(\mathrm{Tr}A_0^2)^2 + \ldots\,, \\
m_{\text{E}}^2 &= g^2T^2(\frac{N_c}{3} + \frac{N_f}{6}) + \ldots, \quad \lambda_{\text{E}} \sim g^4 + \ldots\,.
\end{aligned}
$$

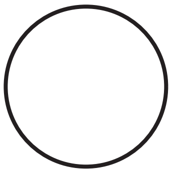
Pressure obtains a factorised form (Λ_{E} is a matching scale):

$$
\begin{aligned}
\frac{p(T)}{T^4} &= 1 + g^2 + g^4 \ln\frac{2\pi T}{\Lambda_{\text{E}}} + g^6 \ln\frac{2\pi T}{\Lambda_{\text{E}}} + \ldots + \frac{p_{\text{E}}(T)}{T^4}\,, \\
p_{\text{E}}(T) &= \lim_{V\to\infty}\frac{T}{V}\ln\int_{\Lambda_{\text{E}}}\mathcal{D}A_i\mathcal{D}A_0\exp(-\frac{1}{T}\int \mathrm{d}^3x\mathcal{L}_{\text{E}})\,.
\end{aligned}
$$

The perturbative expansion for $\mathcal{L}_{\text{E}}$ is, however, slower:

1-loop:

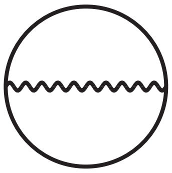
Kapusta 1979



$$p_{\text{E}}(T) \sim T \int \mathrm{d}^3 p \, \ln(p^2 + m_{\text{E}}^2) \sim +g^3 T^4 \, .$$

2-loop:

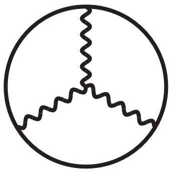
Toimela 1983;
Arnold, Zhai 1994



$$\delta p_{\text{E}}(T) \sim g^4 \ln \frac{\Lambda_{\text{E}}}{m_{\text{E}}} T^4 + g^4 T^4 \, .$$

3-loop:

Zhai, Kastening 1995;
Braaten, Nieto 1995



$$\delta p_{\text{E}}(T) \sim g^5 T^4 \, .$$

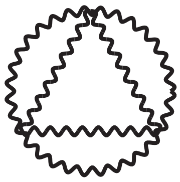
The strange expansion parameter can be understood from

$$\epsilon_b \sim \frac{1}{\pi} g^2 n_b(p) = \frac{g^2}{\pi(e^{p/T} - 1)} \overset{p<T}{\sim} \frac{g^2 T}{\pi p} \, .$$

For hard modes, $p \sim 2\pi T$, expansion is $\sim g^2/\pi^2$ like at $T = 0$, while for colour-electric modes, $p \sim m_{\text{E}} \sim gT$, it is only $\sim g/\pi$.

But colour-magnetic fields remain unscreened, which still causes a logarithmic IR problem at 4-loop level.

Linde 1980;
Gross, Pisarski, Yaffe 1981



$$\sim g^6 T^4 \int \frac{\mathrm{d}^3p \, \mathrm{d}^3q \, \mathrm{d}^3r \, \mathrm{d}^3s}{p^2 q^2 r^2 (s-p)^2 (s-q)^2 (s-r)^2}$$

This can be understood by integrating out A_0.

The effective theory remaining ("Magnetostatic QCD"):

$$\mathcal{L}_{\mathsf{M}} = \frac{1}{2}\mathrm{Tr}\, F_{ij}F_{ij} + \ldots \, .$$

Pressure again obtains a factorised form:

$$\frac{p_{\mathsf{E}}(T)}{T^4} = g^3 + g^4 \ln\frac{\Lambda_{\mathsf{E}}}{m_{\mathsf{E}}} + g^5 + g^6 \ln\frac{\Lambda_{\mathsf{E}}}{m_{\mathsf{E}}} + g^6 \ln\frac{m_{\mathsf{E}}}{\Lambda_{\mathsf{M}}} + \frac{p_{\mathsf{M}}(T)}{T^4},$$

$$p_{\mathsf{M}}(T) = \lim_{V\to\infty}\frac{T}{V}\ln\int_{\Lambda_{\mathsf{M}}}\mathcal{D}A_i\, \exp\Big(-\frac{1}{T}\int \mathrm{d}^3x\, \mathcal{L}_{\mathsf{M}}\Big) \, .$$

The only scale in $\mathcal{L}_{\mathsf{M}}$ is g^2T, and it is a confining theory $\Rightarrow$

$$p_{\mathsf{M}}(T) \sim T \times (g^2T)^3 \times \Big(\ln\frac{\Lambda_{\mathsf{M}}}{g^2T} + [\text{non-pert}]\Big) \, .$$

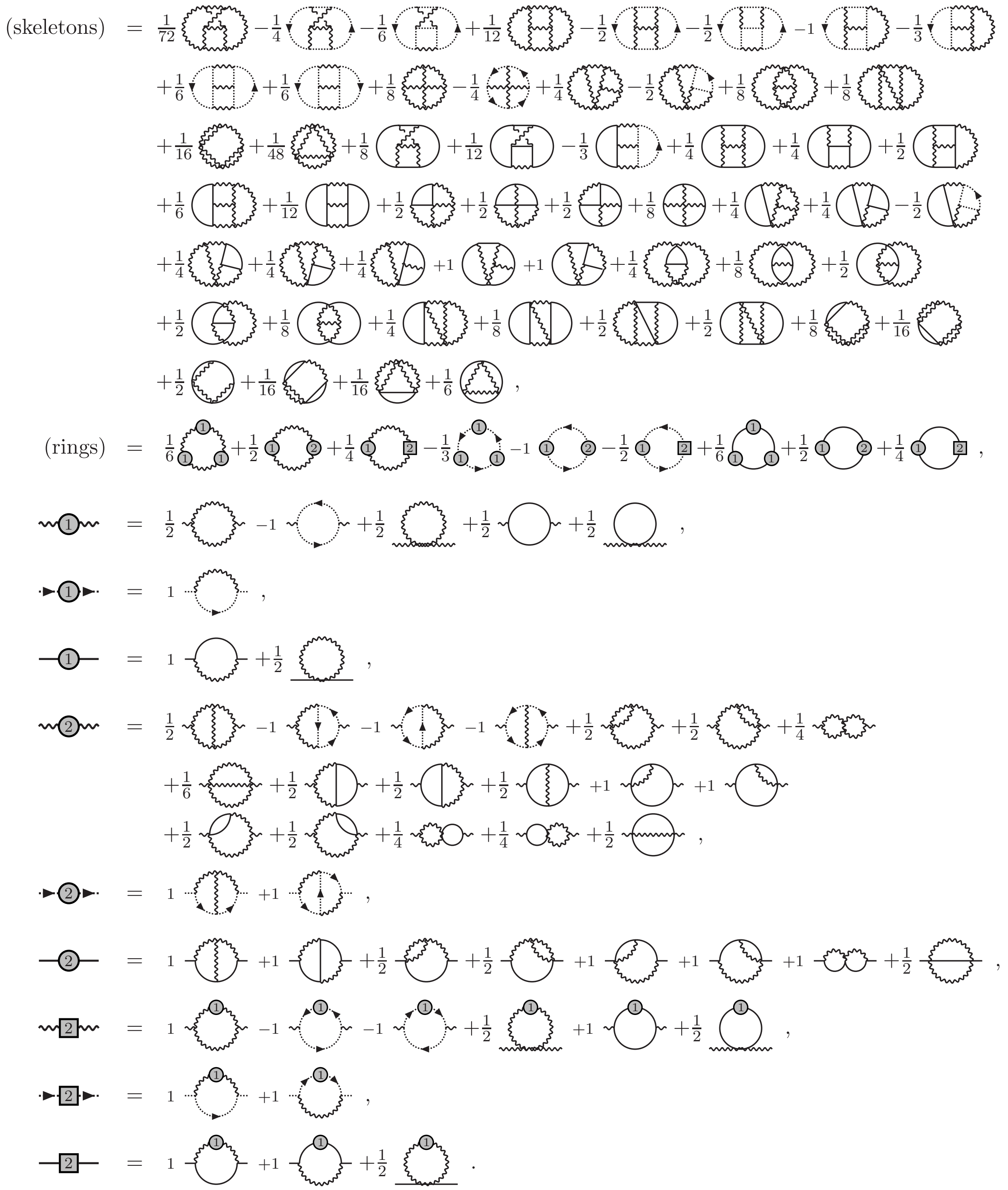
The last thing perturbation theory yields are thus 4-loop logarithms, which originate from a ratio of two physical scales, once the unphysical matching scales Λ_{E}, Λ_{M} cancel:

$$g^6 T^4 \ln \frac{2\pi T}{gT}, \quad g^6 T^4 \ln \frac{gT}{g^2 T} \; .$$

The coefficients can be computed by using $\mathcal{L}_{\mathsf{E}}$, $\mathcal{L}_{\mathsf{M}}$.

Y.Schröder 2002; Kajantie et al 2002

Graphs for the perturbative $g^6 \ln(1/g)\, T^4$ terms:



Non-perturbative IR effects can be related to condensates like

$$\langle \frac{1}{2} \mathrm{Tr}\, F_{ij} F_{ij} \rangle \ ,$$

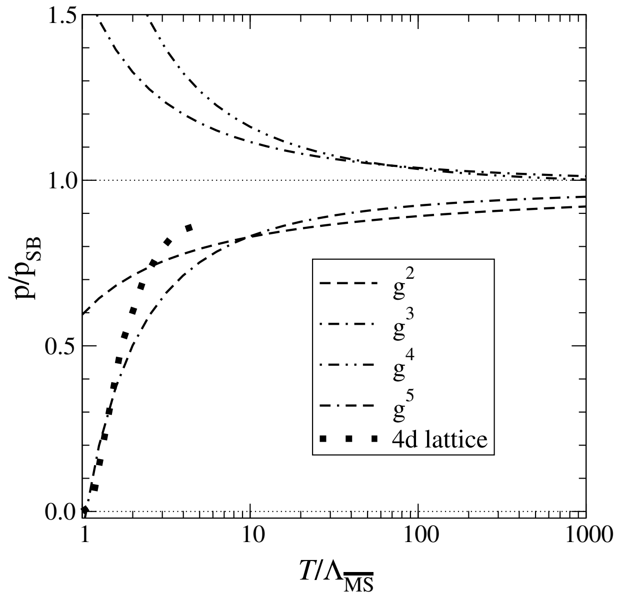
and can be determined on the lattice with $\mathcal{L}_{\mathrm{M}}$.

Braaten, Nieto 1995;
Karsch, Lütgemeier, Patkós, Rank 1996

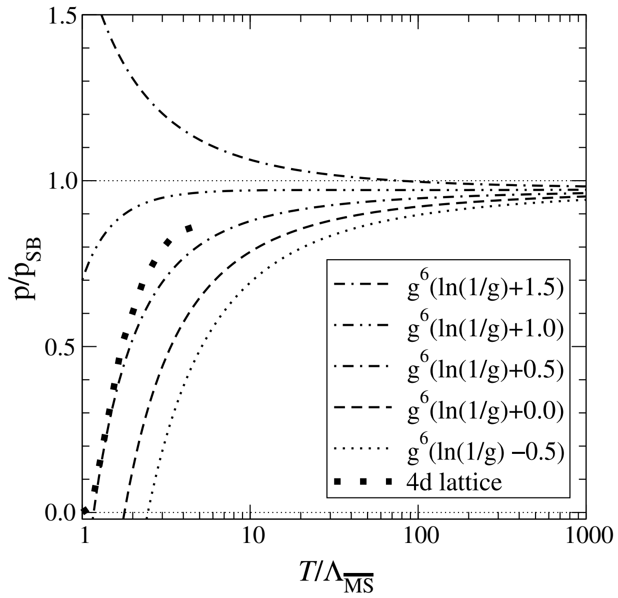
Interpretation of the non-perturbative results requires however a 4-loop matching computation between continuum and lattice regularisation schemes, which has not been finalised yet:

$$\text{[non-pert]} = \langle \frac{1}{2} \mathrm{Tr}\, F_{ij} F_{ij} \rangle - \frac{1}{a^3} - \frac{g^2 T}{a^2} - \frac{(g^2 T)^2}{a}$$
$$- (g^2 T)^3 (\ln \frac{1}{g^2 T a} + \text{[unknown matching coefficient]}) \ .$$

Status (for $N_f = 0$) before inclusion of $g^6[\ln(1/g) + \text{const}]$:



Inclusion of $g^6 \ln(1/g)$ and yet undetermined constant:



Where to go from here?

In principle 4d lattice simulations are sufficient for this problem, although expensive beyond $T \sim (4...5)\Lambda_{\overline{\rm MS}}$ as well as for light dynamical fermions and $0 \lesssim \mu_B \lesssim T$.

It would be disappointing, however, if one could not understand the dynamics of the system even at $T \gg$ GeV.

3d lattice simulations with $\mathcal{L}_{\rm E}$ are OK, but 4-loop matching coefficients are still needed. Extendible to $N_f > 0$ and $\mu_B \lesssim T$ without significant additional cost.

Minimal non-perturbative input is 3d lattice simulations of $\mathcal{L}_{\rm M}$, fixing $\mathcal{O}(g^6T^4)$. Then have to rely, however, on a perturbative treatment of the slowly convergent series from the scale $\sim gT$.

There have also been many analytic attempts at addressing the scales $\sim gT$ by further reorganisations of perturbation theory.

Andersen, Braaten, Petitgirard, Strickland;
Blaizot, Iancu, Rebhan;
Peshier; Parwani; Cvetič, Kögerler; ...

Conclusions

From the theoretical as well as phenomenological point of view, one of the most important observables of finite temperature QCD is minus the free energy density, or pressure, $p(T, \mu_B)$.

It's computation is however plagued by serious IR sensitivity, even at $T \gg$ GeV.

Effective theory methods allow to cure some of these problems perturbatively, and offer a framework for addressing the non-perturbative part in a simplified setting, for $T \gg$ GeV, $\mu_B \lesssim T$.

In the opposite limit $\mu_B \gg 0, T \lesssim \mu_B$, on the other hand, very few first principles results are available up to date.