9.7. Transport coefficients

An important class of observables are so-called transport coefficients. They generically describe the response of the system to some external perturbation. For instance, *electrical conductivity* determines the current that an external electric field leads to. Other standard transport coefficients are diffusion constants related to how an externally induced charge excess in some region flattens out, and shear and bulk viscosities, related to how excesses in momentum flow die away.

We illustrate here some basic aspects of the formalism related to transport coefficients with the example of a *decay rate*. In this case, the external perturbation has displaced the system from equilibrium by giving it a net charge of some type. We assume, however, that the charge is not globally conserved, but that there are processes which violate charge conservation. In this case, the system will *relax back to equilibrium*, i.e. the net charge will disappear, and the decay rate describes how fast this process takes place. The formalism for treating this case can be found in §118, 122, 124, 126 of ref.[48].

Let $\hat{N}(t)$ be the Heisenberg operator of some physical quantity, such as the lepton number. We will consider a situation where the system is out-of-equilibrium in the sense that the expectation value of $\hat{N}(t)$ differs from its equilibrium expectation value, which we assume to be zero:

$$\langle \hat{N}(t) \rangle_{\text{eq}} = 0 \; . \tag{9.174}$$

The non-vanishing non-equilibrium expectation value, $\langle \hat{N}(t) \rangle_{\text{non-eq}}$, is assumed to evolve so slowly that all other reactions are in equilibrium. Then $\langle \hat{N}(t) \rangle_{\text{non-eq}}$ should evolve towards zero, and if $\langle \hat{N}(t) \rangle_{\text{non-eq}}$ is small in some sense [even though it should still be larger than typical thermal fluctuations, $(\langle \hat{N}^2 \rangle_{\text{eq}})^{1/2}$], we can expect the evolution to be described by an equation of first order in $\langle \hat{N}(t) \rangle_{\text{non-eq}}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{N}(t) \rangle_{\text{non-eq}} = -\Gamma \, \langle \hat{N}(t) \rangle_{\text{non-eq}} + \mathcal{O}\Big(\langle \hat{N}(t) \rangle^2_{\text{non-eq}} \Big) \; . \tag{9.175}$$

The coefficient Γ may be called the *decay rate*. The goal would be to obtain an expression for Γ, describing "dissipation", in terms of various *equilibrium* expectation values, of the type $\langle ... \rangle_{\text{eq}}$, describing "fluctuations".

Before proceeding, let us stress that the form of Eq. (9.175) applies to other transport coefficients as well. Consider, as an example, electrical conductivity, σ, defined through

$$\langle \hat{\mathbf{j}} \rangle = \sigma \mathbf{E} \; , \tag{9.176}$$

where $\mathbf{E}$ is an external electric field. Taking a time derivative leads to

$$\frac{\partial \langle \hat{\mathbf{j}} \rangle}{\partial t} = \sigma \frac{\partial \mathbf{E}}{\partial t} = -\sigma \langle \hat{\mathbf{j}} \rangle \; , \tag{9.177}$$

where we used the Maxwell equation $\nabla \times \mathbf{B} - \partial \mathbf{E}/\partial t = \langle \hat{\mathbf{j}} \rangle$, and assumed that the external probe field $\mathbf{B}$ has been set to zero. Clearly Eq. (9.177) has precisely the form of Eq. (9.175).

In order derive an equation of the type in Eq. (9.175), let us write a generic linear response formula for $\langle \hat{N}(t) \rangle_{\text{non-eq}}$. We can assume that at time $t = -\infty$, the system was in full equilibrium, but then a source term was added to the Hamiltonian,

$$\hat{H}(t) = \hat{H}_0 - \mu(t) \hat{N}(t) \; , \tag{9.178}$$

which will slowly displace $\langle \hat{N}(t) \rangle_{\text{non-eq}}$ from zero. Solving the equation of motion for the density matrix $\hat{\rho}(t)$,

$$i \frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = \Big[\hat{H}(t), \hat{\rho}(t) \Big] \; , \tag{9.179}$$

[48] L.D. Landau and E.M. Lifshitz, *Statistical Physics*, Part 1, 3rd Edition (Pergamon Press, Oxford, 1993).

to first order in the perturbation,

$$\hat{\rho}(t) \approx \hat{\rho}(-\infty) - i \int_{-\infty}^{t} \mathrm{d}t' \left[\hat{H}(t'), \hat{\rho}(-\infty) \right] + \ldots \,, \tag{9.180}$$

yields

$$\begin{aligned} \langle \hat{N}(t) \rangle_{\text{non-eq}} &= \mathrm{Tr}\,[\hat{\rho}(t)\hat{N}(t)] \\ &\approx i \int_{-\infty}^{t} \mathrm{d}t'\, \mu(t') \mathrm{Tr} \left\{ \left[\hat{N}(t'), \hat{\rho}(-\infty) \right] \hat{N}(t) \right\} \\ &= i \int_{-\infty}^{\infty} \mathrm{d}t' \left\langle [\hat{N}(t), \hat{N}(t')] \right\rangle_{\text{eq}} \theta(t-t')\, \mu(t') \,, \end{aligned} \tag{9.181}$$

where

$$\langle ... \rangle_{\text{non-eq}} \equiv \mathrm{Tr}\,[\hat{\rho}(t)(...)] \,, \qquad \langle ... \rangle_{\text{eq}} \equiv \mathrm{Tr}\,[\hat{\rho}(-\infty)(...)] \,, \tag{9.182}$$

and the leading term disappeared because of the assumption in Eq. (9.174). We also made use of the fact that $\hat{H}_0$ and $\hat{\rho}(-\infty)$ commute. In addition, we inserted $\theta(t-t')$ and extended afterwards the upper end of the integration to infinity, to stress the retarded nature of the correlator. The equilibrium expectation value in Eq. (9.181) is called a *linear response function*.

Eq. (9.181) is now in principle of a type that could be used in Eq. (9.175), expect that it contains the function $\mu(t')$ on the right-hand side, rather the quantity $\langle \hat{N}(t') \rangle_{\text{non-eq}}$ which would be needed for indentifying Γ. In some cases further input may allow to establish a bridge between $\mu(t')$ and $\langle \hat{N}(t') \rangle_{\text{non-eq}}$, but below we choose to follow another approach. However, as we will see, Eq. (9.181) is still quite useful, because it shows explicitly that the essential information should reside in the retarded Green's function $\langle [\hat{N}(t), \hat{N}(t')] \rangle_{\text{eq}} \theta(t-t')$.

Indeed, following §118 of ref.[48], let us define the correlator

$$\Delta(t) \equiv \left\langle \frac{1}{2} \left\{ \hat{N}(t), \hat{N}(0) \right\} \right\rangle_{\text{eq}} . \tag{9.183}$$

The value $\Delta(0)$ amounts to a "susceptibility" and is trivial to compute perturbatively; for the number operator of a massless chiral fermion, for instance, one gets by taking the second derivative from Eq. (7.41), and dividing by two because we are considering Majorana rather than Dirac fermions,

$$\begin{aligned} \Delta(0) = \langle \hat{N}^2 \rangle_{\text{eq}} &= \frac{1}{\mathcal{Z}} T^2 \partial_\mu^2 \mathrm{Tr} \left[e^{-\beta(\hat{H}_0 - \mu \hat{N})} \right]_{\mu=0} \\ &= T^2 \Big[\partial_\mu^2 \ln \mathcal{Z}(T,\mu) \Big]_{\mu=0} \\ &= -T^2 \Big[\partial_\mu^2 \frac{f(T,\mu)V}{T} \Big]_{\mu=0} \\ &= \frac{1}{6} V T^3 \,, \end{aligned} \tag{9.184}$$

where V is the volume. Furthermore, it is clear that $\Delta(-t) = \Delta(t)$ (by time-translational invariance). Finally, at large time separations the fluctuations are uncorrelated, so that

$$\lim_{t \to \infty} \Delta(t) = 0 \,. \tag{9.185}$$

Landau and Lifshitz now argue that the way $\Delta(t)$ approaches zero (on time scales much larger than the ones needed to reach equilibrium for any given non-zero $\langle \hat{N}(t) \rangle_{\text{non-eq}}$, i.e. certainly $t \gg 1/T$) is determined precisely by the coefficient Γ in Eq. (9.175). We recall that according to Eq. (9.181), Γ must in any case be related to equilibrium expectation values. Concretely, the behaviour should be[48]

$$\Delta(t) \simeq \Delta(0)\, \exp(-\Gamma |t|) \,. \tag{9.186}$$

Let us at this stage define several auxiliary objects (cf. Sec. 8.1): a Fourier-transform of $\Delta(t)$,

$$\tilde{\Delta}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, \Delta(t) \; ; \tag{9.187}$$

a retarded correlator,

$$C_R(t) \equiv \Big\langle i \Big[\hat{N}(t), \hat{N}(0)\Big] \, \theta(t) \Big\rangle_{\mathrm{eq}} \; ; \tag{9.188}$$

and its Fourier-transform,

$$\tilde{C}_R(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, C_R(t) \; . \tag{9.189}$$

Standard thermodynamic relations (Eq. (8.17)) dictate that

$$\tilde{\Delta}(\omega) = \Big[1 + 2n_{\mathrm{B}}(\omega)\Big] \rho(\omega) \tag{9.190}$$

$$= \Big[1 + 2n_{\mathrm{B}}(\omega)\Big] \frac{1}{2i} \Big[\tilde{C}_R(\omega + i0^+) - \tilde{C}_R(\omega - i0^+)\Big] \; . \tag{9.191}$$

Carrying out the Fourier integral on Eq. (9.186) implies that, for small frequencies,

$$\tilde{\Delta}(\omega) \simeq \Delta(0) \bigg[\int_{-\infty}^{0} \mathrm{d}t \, e^{(i\omega + \Gamma)t} + \int_{0}^{\infty} \mathrm{d}t \, e^{(i\omega - \Gamma)t} \bigg]$$

$$= \Delta(0) \frac{2\Gamma}{\omega^2 + \Gamma^2} \; . \tag{9.192}$$

"Small frequencies" certainly means $\omega \ll T$; in fact, taking $\omega \to 0$ on both sides of Eq. (9.192) and using Eq. (9.190) yields

$$\frac{1}{\Gamma} \Delta(0) = \lim_{\omega \to 0} \frac{T}{\omega} \rho(\omega) \; . \tag{9.193}$$

We recall that $\Delta(0)$ here is given by Eq. (9.184), so that Eq. (9.193) in principle allows to determine Γ from the slope of the spectral function at zero frequency (which in turn is related to the retarded correlator, as we expected from Eq. (9.181)). This is an example of a *Kubo formula*.

Eq. (9.193) looks actually somewhat suspicious. Indeed, consider the free limit. In this case, the decay rate Γ should disappear. Therefore, the right-hand side should diverge. However, according to what we saw for the free spectral functions of single particles states (cf. Eqs. (8.36), (8.66)), we would normally expect the spectral function to be *zero* at small enough frequencies ($|\omega| < m$), in which case the right-hand side would vanish.

The resolution to the paradox is that in the free limit the spectral function related to a *conserved charge* contains a contribution of the type

$$\rho(\omega) \simeq \Delta(0) \beta\omega \, \pi \, \delta(\omega) \tag{9.194}$$

$$= \Delta(0) \beta\omega \; \mathrm{Im} \frac{1}{\omega - i0^+} \; . \tag{9.195}$$

Indeed, plugging this into the sum rule in Eq. (8.33), we get

$$\Big\langle \hat{N}(\tau) \hat{N}(0) \Big\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \rho(\omega) n_{\mathrm{B}}(\omega) e^{(\beta - \tau)\omega} = \Delta(0) \; , \tag{9.196}$$

which shows that Eq. (9.194) corresponds to a Euclidean correlator which is contant in τ. This, indeed, is precisely what we expect from the correlator of a conserved quantity! Plugging Eq. (9.194) into the right-hand side of Eq. (9.193) we indeed get infinity, so everything is in order after all.

Now, the structure in Eq. (9.194) is an extreme case of what is called a *transport peak* in the spectral function. Once interactions are added, the peak should get smeared, so that the limit in Eq. (9.193) is finite; in fact, according to Eqs. (9.190), (9.192), it should obtain the form ($\omega \ll T$)

$$\rho(\omega) \simeq \Delta(0)\frac{\beta\omega\Gamma}{\omega^2+\Gamma^2} \tag{9.197}$$

$$= \Delta(0)\beta\omega\,\mathrm{Im}\,\frac{1}{\omega - i\Gamma}\,. \tag{9.198}$$

Therefore, in order to identify the transport coefficients, we need to be able to compute the spectral function in the range $\omega \sim \Gamma$. This is typically difficult because Γ is generated by interactions, and therefore of the type $\Gamma \sim g^2 T$, where g is some coupling constant: we need to be able to compute $\rho(\omega)$ for soft energies $\omega \sim g^2 T$, where resummations play an important role. In any case, if the computation is successful, the resulting structure should be of the type

$$\rho(\omega) \simeq \Delta(0)\frac{\beta\Gamma}{\omega} + \mathcal{O}(\Gamma^3)\,, \tag{9.199}$$

which replaces the vanishing result at non-zero ω that was obtained in the free limit, Eq. (9.194).