9.5. Dark matter abundance in cosmology

In this section we continue the considerations of Sec. 9.4. Denoting by $n(t,\mathbf{q})$ the phase space density of right-handed neutrinos in either polarization state,

$$n(t,\mathbf{q}) = \sum_{s=1,2} \frac{\mathrm{d}N^{(s)}(t,\mathbf{x},\mathbf{q})}{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}} \;, \tag{9.126}$$

Eq. (9.96) is of the form

$$\frac{\partial n(t,\mathbf{q})}{\partial t} = R(T,\mathbf{q}) \;, \tag{9.127}$$

where R is given in Eq. (9.125),

$$R(T,\mathbf{q}) = \frac{2n_\mathrm{F}(q^0)}{(2\pi)^3 2q^0} \sum_{\alpha=1}^{3} |F_\alpha|^2 \mathrm{Tr}\Big\{ \not{Q}\, a_L \Big[\rho_{\alpha\alpha}(-Q) + \rho_{\alpha\alpha}(Q) \Big] a_R \Big\} \;. \tag{9.128}$$

This applies in flat spacetime, at a given temperature T. Note that in thermal equilibrium, in the free limit, we would expect $n(t,q) = 2n_\mathrm{F}(q^0)/(2\pi)^3$, where the factor 2 comes the polarization sum, and the factor $(2\pi)^3$ from that according to Eq. (9.126), $\sum_{s=1,2} N^{(s)} = \int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}\, n(t,\mathbf{q})$. For cosmological applications, the first task now is to generalize Eq. (9.127) to an expanding Universe, where the temperature furthermore is a function of time.

In order to carry out the generalization, we first have to give a meaning to our variables, the time t and the momenta $\mathbf{q}$. In the following we mean by these the *physical* time and momenta, i.e. the ones defined in a local Minkowskian frame. However, as is well known, local Minkowskian frames at different times are inequivalent in an expanding background; in particular, the physical momenta redshift. Carrying out the derivation of the rate equation in this situation is a topic of general relativity rather than thermal field theory, so we only quote the result here: the upshot is that the time derivative gets replaced with $\partial/\partial t \to \partial/\partial t - Hq_i\partial/\partial q_i$[46], and Eq. (9.125) becomes

$$\Big(\frac{\partial}{\partial t} - Hq_i \frac{\partial}{\partial q_i} \Big) n(t,\mathbf{q}) = R(T,\mathbf{q}) \;, \tag{9.129}$$

where H is the Hubble parameter, $H(t) \equiv \dot{a}(t)/a(t)$, and q_i are the spatial components of $\mathbf{q}$. We will see presently that this form is in any case *consistent* with the expected redshift $\mathbf{q}(t) = \mathbf{q}(t_0)\, a(t_0)/a(t)$, where $a(t)$ is the scale factor.

Eq. (9.129) can be written in a simpler form through a change of variables. We note, first of all, that $R(T,\mathbf{q})$ and consequently also $n(t,\mathbf{q})$ are only functions of $q \equiv |\mathbf{q}|$. Changing correspondingly the notation to $n(t,q)$, $R(T,q)$, Eq. (9.129) becomes

$$\Big(\frac{\partial}{\partial t} - Hq \frac{\partial}{\partial q} \Big) n(t,q) = R(T,q) \;. \tag{9.130}$$

Introducing then an ansatz $n(t,q) = n(t, q(t_0)\frac{a(t_0)}{a(t)})$, and noting that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big[q(t_0) \frac{a(t_0)}{a(t)} \Big] = -q(t_0) \frac{a(t_0)\dot{a}(t)}{a^2(t)} = -Hq \;, \tag{9.131}$$

Eq. (9.130) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} n\Big(t, q(t_0)\frac{a(t_0)}{a(t)}\Big) = R\Big(T, q(t_0)\frac{a(t_0)}{a(t)}\Big) \;. \tag{9.132}$$

[46] J. Bernstein, *Kinetic Theory in the Expanding Universe* (Cambridge University Press, Cambridge, 1988); E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Reading, 1990).

This integrates immediately to

$$n(t_0, q(t_0)) = \int_0^{t_0} \mathrm{d}t\, R\Big(T(t), q(t_0)\frac{a(t_0)}{a(t)}\Big) \; , \tag{9.133}$$

where we assumed the initial condition $n(0, q) = 0$, i.e., that there are no right-handed neutrinos in the beginning.

At this point we need to recall the basic cosmological relations between the time t and the temperature T. Assuming a homogeneous and isotropic metric,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\mathbf{x}_k^2 \; , \tag{9.134}$$

and the energy-momentum tensor of an ideal fluid,

$$T_\mu{}^\nu = \mathrm{diag}(e, -p, -p, -p) \; , \tag{9.135}$$

where e denotes the energy density and p the pressure, the Einstein equations, $G_\mu{}^\nu = 8\pi G T_\mu{}^\nu$, reduce to

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} e \; , \tag{9.136}$$

$$\mathrm{d}(ea^3) = -p\,\mathrm{d}(a^3) \; . \tag{9.137}$$

We will assume a flat Universe in the following, $k = 0$, and denote

$$\frac{1}{m_{\mathrm{Pl}}^2} \equiv G \; , \tag{9.138}$$

where $m_{\mathrm{Pl}} = 1.2 \times 10^{19}$ GeV is the Planck mass.

We now combine the Einstein equations with basic thermodynamic relations. Assuming a system with at most very small chemical potentials, the energy and entropy densities are related by

$$e = Ts - p \; , \tag{9.139}$$

where $s = \mathrm{d}p/\mathrm{d}T$ is the entropy density. The derivative of Eq. (9.139) with respect to T yields

$$\frac{\mathrm{d}e}{\mathrm{d}T} = T\frac{\mathrm{d}s}{\mathrm{d}T} = Tc \; , \tag{9.140}$$

where c is the heat capacity. Moving all the terms in Eq. (9.137) to the left-hand side, we get

$$a^3\mathrm{d}e + 3(e+p)a^2\mathrm{d}a = 0 \tag{9.141}$$

$$\Leftrightarrow \quad \mathrm{d}e = -3Ts\frac{\mathrm{d}a}{a} \tag{9.142}$$

$$\Leftrightarrow \quad \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\mathrm{d}e}{\mathrm{d}T} = -3Ts\frac{\dot{a}}{a} \; . \tag{9.143}$$

Inserting Eqs. (9.136), (9.138) and (9.140) results finally in

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\sqrt{24\pi}}{m_{\mathrm{Pl}}}\frac{s(T)\sqrt{e(T)}}{c(T)} \; . \tag{9.144}$$

In addition, Eq. (9.137) can also be written in terms of the well-known entropy conservation law:

$$\begin{aligned} 0 &= \mathrm{d}(ea^3) + p\,\mathrm{d}(a^3) \\ &= \mathrm{d}([e+p]a^3) - a^3\mathrm{d}p \\ &= \mathrm{d}(Tsa^3) - a^3 s\mathrm{d}T \\ &= T\mathrm{d}(sa^3) \; , \end{aligned} \tag{9.145}$$

where we inserted Eq. (9.139) and the definition $s = \mathrm{d}p/\mathrm{d}T$. Eq. (9.145) can in turn be expressed as

$$\frac{a(t)}{a(t_0)} = \left[\frac{s(T_0)}{s(T)}\right]^{\frac{1}{3}} \,, \tag{9.146}$$

where t and T are related through Eq. (9.144).

It is furthermore conventional to introduce the effective numbers of massless bosonic degrees of freedom $g_{\rm eff}(T)$ and $h_{\rm eff}(T)$ via the relations

$$e(T) \equiv \frac{\pi^2 T^4}{30} g_{\rm eff}(T)\,, \quad s(T) \equiv \frac{2\pi^2 T^3}{45} h_{\rm eff}(T)\,, \tag{9.147}$$

where the prefactors follow by applying Eq. (9.139) and the line below it to the free result $p(T) = \pi^2 T^4/90$ from Eq. (2.80). Given the equation-of-state of the plasma [i.e. the relation between the pressure and the temperature, $p = p(T)$], $g_{\rm eff}(T)$ and $h_{\rm eff}(T)$ can according to Eq. (9.139) be found from

$$g_{\rm eff}(T) = \frac{30}{\pi^2 T^2}\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{p}{T}\right)\,, \quad h_{\rm eff}(T) = \frac{45}{2\pi^2 T^3}\frac{\mathrm{d}p}{\mathrm{d}T}\,. \tag{9.148}$$

Furthermore, we note that $s(T)/Tc(T) = p'(T)/Ts'(T) = p'(T)/e'(T) = \partial p/\partial e = c_s^2(T)$, where we identified the standard expression for the sound speed squared.

Combining Eqs. (9.144), (9.146) and the definitions above, we can now change variables in Eq. (9.133), arriving at ($q \equiv q(t_0)$)

$$n(t_0, q) = \int_{T_0}^{\infty} \frac{\mathrm{d}T}{T^3}\sqrt{\frac{5}{4\pi^3}}\frac{m_{\rm Pl}}{c_s^2(T)\sqrt{g_{\rm eff}(T)}} R\biggl(T, q\frac{T}{T_0}\left[\frac{h_{\rm eff}(T)}{h_{\rm eff}(T_0)}\right]^{\frac{1}{3}}\biggr)\,. \tag{9.149}$$

The integral over Eq. (9.149) with the measure $\mathrm{d}^3\mathbf{q}$ gives the number density of right-handed neutrinos, denoted by $n(t_0)$, which can subsequently be conveniently normalized with respect to the total entropy density, Eq. (9.147), which produces the so-called *yield parameter*, Y:

$$\begin{aligned} Y(t_0) &\equiv \frac{n(t_0)}{s(t_0)} \\ &= \frac{45\times 4\pi}{2\pi^2 T_0^3 h_{\rm eff}(T_0)}\sqrt{\frac{5}{4\pi^3}}\int_0^\infty \mathrm{d}q\, q^2 \int_{T_0}^\infty \frac{\mathrm{d}T}{T^3}\frac{m_{\rm Pl}}{c_s^2(T)\sqrt{g_{\rm eff}(T)}} R\biggl(T, T\frac{q}{T_0}\left[\frac{h_{\rm eff}(T)}{h_{\rm eff}(T_0)}\right]^{\frac{1}{3}}\biggr) \\ &= \frac{45\sqrt{5}}{\pi^{5/2}}\int_{T_0}^\infty \frac{\mathrm{d}T}{T^3}\int_0^\infty \mathrm{d}z\, z^2\, \frac{m_{\rm Pl}}{c_s^2(T) h_{\rm eff}(T)\sqrt{g_{\rm eff}(T)}} R\left(T, Tz\right)\,, \end{aligned} \tag{9.150}$$

where in the last step we substituted $q = zT_0[h_{\rm eff}(T_0)/h_{\rm eff}(T)]^{1/3}$. Note that Eq. (9.150) obtains a constant value at low temperatures if $R \to 0$. Therefore the yield parameter is a good (i.e. T_0-independent) characterization of the dark matter relic density.

In order to write the result more explicitly, we need to specify the function $R(T, q)$. The relevant information is contained in Eqs. (8.88), (8.89), (9.128). The Dirac algebra in Eq. (9.128) can be carried out:

$$\mathrm{Tr}\left\{ \not{Q}\, a_L \Bigl[\not{P}_1\Bigr] a_R \right\} = 2Q\cdot P_1\,. \tag{9.151}$$

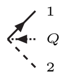
Furthermore, this can be written in various ways depending on the channel:

$$\begin{aligned} \delta^{(4)}(P_1+P_2-Q)\, 2Q\cdot P_1 &= \delta^{(4)}(P_1+P_2-Q)[P_1^2+Q^2-(Q-P_1)^2] \\ &= \delta^{(4)}(P_1+P_2-Q)[P_1^2+Q^2-P_2^2]\,, \\ \delta^{(4)}(P_2-P_1-Q)\, 2Q\cdot P_1 &= \delta^{(4)}(P_2-P_1-Q)[(Q+P_1)^2-P_1^2-Q^2] \\ &= \delta^{(4)}(P_2-P_1-Q)[P_2^2-P_1^2-Q^2]\,, \\ \delta^{(4)}(P_1-P_2-Q)\, 2Q\cdot P_1 &= \delta^{(4)}(P_1-P_2-Q)[P_1^2+Q^2-(Q-P_1)^2] \end{aligned}$$

$$
\begin{aligned}
&= \delta^{(4)}(P_1 - P_2 - Q)[P_1^2 + Q^2 - P_2^2] \,, \\
\delta^{(4)}(P_1 + P_2 + Q)\, 2Q\cdot P_1 &= \delta^{(4)}(P_1 + P_2 + Q)[(Q+P_1)^2 - P_1^2 - Q^2] \\
&= \delta^{(4)}(P_1 + P_2 + Q)[P_2^2 - P_1^2 - Q^2] \,. \qquad (9.152)
\end{aligned}
$$

These factors are all constants, independent of the momenta $\mathbf{p}_1, \mathbf{p}_2$. Thereby we arrive at

$$
\begin{aligned}
R(T,\mathbf{q}) = \frac{1}{(2\pi)^3 2q^0} \sum_{\alpha=1}^{3} |F_\alpha|^2 \sum_c (m_{\phi_c}^2 - m_{\ell_c}^2 - M^2) \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \times \\
\times \Big\{ \; - (2\pi)^4 \delta^{(4)}(P_1 + P_2 - Q)\, n_{\mathrm{F1}} n_{\mathrm{B2}} + \\
+ (2\pi)^4 \delta^{(4)}(P_2 - P_1 - Q)\, n_{\mathrm{B2}} (1 - n_{\mathrm{F1}}) + \\
- (2\pi)^4 \delta^{(4)}(P_1 - P_2 - Q)\, n_{\mathrm{F1}} (1 + n_{\mathrm{B2}}) + \\
+ (2\pi)^4 \delta^{(4)}(P_1 + P_2 + Q)\, (1 - n_{\mathrm{F1}})(1 + n_{\mathrm{B2}}) \Big\} \,, \qquad (9.153)
\end{aligned}
$$



where $E_1 \equiv \sqrt{m_{\ell_c}^2 + \mathbf{p}^2}$, $E_2 \equiv \sqrt{m_{\phi_c}^2 + (\mathbf{p}+\mathbf{q})^2}$. In passing, it is interesting to note that Eq. (9.153) resembles very much a *collision term of a Boltzmann equation*. We briefly recall the structure of the latter in Sec. 9.6.

Let us analyse Eq. (9.153) more precisely. Suppose that $q^0 > 0$. The first question is, when do the different channels get realized? Since all particles are massive, we can go to the rest frame of one of them; it is then clear that the first channel gets realized for $M > m_{\ell_c} + m_{\phi_c}$; the second for $m_{\phi_c} > M + m_{\ell_c}$; the third for $m_{\ell_c} > M + m_{\phi_c}$; and the last one never. So, assuming that the scalar mass (the Higgs mass) is larger than those of the produced particles, $m_{\phi_c} \gg M, m_{\ell_c}$, we can focus on the second channel, and the integral to be considered is

$$
I \equiv \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(P_2 - P_1 - Q)\, n_{\mathrm{B}}(u\cdot P_2)[1 - n_{\mathrm{F}}(u\cdot P_1)] \,. \qquad (9.154)
$$

Here u is the four-velocity of the thermal bath, and we have written the integral in a frame-independent way.

Remarkably, the integral in Eq. (9.154) can be written in a very simple form, in the high-temperature limit where the masses $M^2 = Q^2$ and $m_{\ell_c}^2 = P_1^2$ of the produced particles can be neglected[47]. Denoting

$$
p \equiv |\mathbf{p}_1| \,, \quad q \equiv |\mathbf{q}| \,, \qquad (9.155)
$$

we get

$$
\begin{aligned}
I &= \int \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2p} \int \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{q} - \mathbf{p}_2)\, (2\pi)\delta(p + q - E_2) n_{\mathrm{B}}(E_2)[1 - n_{\mathrm{F}}(p)] \\
&= \frac{1}{(4\pi)^2} \int \frac{\mathrm{d}^3\mathbf{p}_1}{p(p+q)} \delta\Big(p + q - \sqrt{m_{\phi_c}^2 + (\mathbf{p}_1 + \mathbf{q})^2}\Big) n_{\mathrm{B}}(p+q)[1 - n_{\mathrm{F}}(p)] \\
&= \frac{1}{8\pi} \int_0^\infty \frac{\mathrm{d}p\, p}{p+q} \int_{-1}^{+1} \mathrm{d}z\, \delta\Big(p + q - \sqrt{m_{\phi_c}^2 + p^2 + q^2 + 2pqz}\Big) n_{\mathrm{B}}(p+q)[1 - n_{\mathrm{F}}(p)] \,, \quad (9.156)
\end{aligned}
$$

where in the last step radial coordinates were introduced, with $\mathbf{q}$ as the z-axis. The delta function gets realized when

$$
p^2 + q^2 + 2pq = m_{\phi_c}^2 + p^2 + q^2 + 2pqz \,, \qquad (9.157)
$$

[47]M. Shaposhnikov, unpublished notes.

i.e. $z = 1 - m_{\phi_c}^2/2pq$. This belongs to the interval $(-1,1)$ if $p > m_{\phi_c}^2/4q$, so that

$$I = \frac{1}{8\pi}\int_{m_{\phi_c}^2/4q}^{\infty} \frac{\mathrm{d}p\,p}{p+q}\left|\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{m_{\phi_c}^2 + p^2 + q^2 + 2pqz}\right|^{-1}_{\sqrt{m_{\phi_c}^2+p^2+q^2+2pqz}=p+q} n_{\mathrm{B}}(p+q)[1-n_{\mathrm{F}}(p)] \,. \tag{9.158}$$

The derivative is trivial,

$$\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{\cdots}\Big|_{\ldots} = \frac{pq}{p+q} \,, \tag{9.159}$$

whereby we finally arrive at

$$\begin{aligned} I &= \frac{1}{8\pi q}\int_{m_{\phi_c}^2/4q}^{\infty} \mathrm{d}p\, n_{\mathrm{B}}(p+q)[1-n_{\mathrm{F}}(p)] \\ &= \frac{T}{8\pi q}\int_0^\infty \mathrm{d}x\, n_{\mathrm{B}}\Big(Tx + q + \frac{m_{\phi_c}^2}{4q}\Big)\Big[1 - n_{\mathrm{F}}\Big(Tx + \frac{m_{\phi_c}^2}{4q}\Big)\Big] \,. \end{aligned} \tag{9.160}$$

In the last step we substituted $p = m_{\phi_c}^2/4q + Tx$.

We can easily work out an *upper bound* for this integral if we make the further approximation that $T \ll m_{\phi_c}$. Indeed, $1 - n_{\mathrm{F}} \le 1$, and

$$q + \frac{m_{\phi_c}^2}{4q} = m_{\phi_c} + \frac{2}{m_{\phi_c}}\Big(q - \frac{m_{\phi_c}}{2}\Big)^2 + \ldots \ge m_{\phi_c} \,. \tag{9.161}$$

Thereby the distribution function n_{B} can to a good approximation be replaced by the Boltzmann distribution, whereby

$$I \lesssim \frac{T}{8\pi q}\int_0^\infty \mathrm{d}x\, e^{-x-\beta\left(q+\frac{m_{\phi_c}^2}{4q}\right)} = \frac{T}{8\pi q}e^{-\beta\left(q+\frac{m_{\phi_c}^2}{4q}\right)} \,. \tag{9.162}$$

Inserting Eq. (9.162) into Eq. (9.153), we get

$$R(T,q) \lesssim \frac{1}{(2\pi)^3 2q}\sum_{\alpha=1}^{3}|F_\alpha|^2 \sum_c m_{\phi_c}^2 \frac{T}{8\pi q}e^{-\beta\left(q+\frac{m_{\phi_c}^2}{4q}\right)} \,. \tag{9.163}$$

Combining now with Eq. (9.150) produces

$$\begin{aligned} Y(t_0) &\lesssim \frac{45\sqrt{5}}{\pi^{5/2}}\frac{1}{128\pi^4}\sum_{\alpha=1}^{3}|F_\alpha|^2\sum_c m_{\phi_c}^2\int_{T_0}^{\infty}\frac{\mathrm{d}T}{T^2}\int_0^\infty \frac{\mathrm{d}z\,z^2}{T^2 z^2}\frac{m_{\mathrm{Pl}}}{c_s^2(T)h_{\mathrm{eff}}(T)\sqrt{g_{\mathrm{eff}}(T)}}e^{-\left(z+\frac{m_{\phi_c}^2}{4T^2 z}\right)} \\ &= \frac{45\sqrt{5}}{64\pi^{13/2}}\sum_{\alpha=1}^{3}|F_\alpha|^2\sum_c m_{\phi_c}^2\int_{T_0}^{\infty}\frac{\mathrm{d}T}{T^4}\frac{m_{\mathrm{Pl}}}{c_s^2(T)h_{\mathrm{eff}}(T)\sqrt{g_{\mathrm{eff}}(T)}}K_1\Big(\frac{m_{\phi_c}}{T}\Big) \,, \end{aligned} \tag{9.164}$$

where K_1 is a Bessel function. For $m_{\phi_c}/T \gg 1$, we can finally approximate

$$K_1\Big(\frac{m_{\phi_c}}{T}\Big) \approx \sqrt{\frac{\pi T}{2m_{\phi_c}}}e^{-\beta m_{\phi_c}} \,, \tag{9.165}$$

and if we approximate c_s^2, h_{eff} and g_{eff} to be slowly varying, the remaining integral can be carried out:

$$\begin{aligned} \int_{T_0}^{\infty}\frac{\mathrm{d}T}{T^4}f(T)\sqrt{\frac{\pi T}{2m_{\phi_c}}}e^{-\beta m_{\phi_c}} &\overset{T=m_{\phi_c}x}{\approx} \frac{1}{m_{\phi_c}^3}\sqrt{\frac{\pi}{2}}\int_0^\infty \mathrm{d}x\, x^{-7/2}f(m_{\phi_c}x)e^{-1/x} \\ &\overset{y=1/x}{=} \frac{1}{m_{\phi_c}^3}\sqrt{\frac{\pi}{2}}\int_0^\infty \mathrm{d}y\, y^{3/2}f\Big(\frac{m_{\phi_c}}{y}\Big)e^{-y} \\ &\simeq \frac{1}{m_{\phi_c}^3}\sqrt{\frac{\pi}{2}}\Gamma\Big(\frac{5}{2}\Big)f\Big(\frac{m_{\phi_c}}{2}\Big) \,, \end{aligned} \tag{9.166}$$

where we replaced y in the argument of f by the value where the integral has obtained about 50% of its total magnitude. Thereby most of the dark matter abundance is indeed generated at temperatures $T \lesssim m_{\phi_c}$, and

$$Y(t_0) \lesssim \frac{135\sqrt{5}}{256\sqrt{2}\pi^{11/2}} \sum_{\alpha=1}^{3} |F_\alpha|^2 \sum_c \frac{m_{\rm Pl}}{m_{\phi_c}} \frac{1}{c_s^2\left(\frac{m_{\phi_c}}{2}\right) h_{\rm eff}\left(\frac{m_{\phi_c}}{2}\right) \sqrt{g_{\rm eff}\left(\frac{m_{\phi_c}}{2}\right)}} , \tag{9.167}$$

where we inserted $\Gamma(5/2) = 3\sqrt{\pi}/4$.

Eq. (9.167) displays clearly the variables on which the dark matter abundance related to a specific mechanism depends on: the coupling constants ($|F_\alpha|^2$), the mass of the decaying particle (m_{ϕ_c}), as well as the thermal history of the Universe (the functions c_s^2, $h_{\rm eff}$, $g_{\rm eff}$).

9.6. Appendix: relativistic Boltzmann equation

We recall here briefly the structure of the collision terms of a relativistic Boltzmann equation, and compare the result with the quantum field theoretic formula in Eq. (9.153).

To understand the logic of the Boltzmann equation, a possible starting point is Fermi's Golden Rule for a decay rate:

$$\Gamma_{1\to n}(Q) = \frac{c}{2E_{\mathbf{q}}}\int \mathrm{d}\Phi_{1\to n}\,|\mathcal{M}_{1\to n}|^2 \;, \tag{9.168}$$

where the phase space integration measure is defined as

$$\int \mathrm{d}\Phi_{1+m\to n} \;\equiv\; \int \left\{\prod_{a=1}^{m} \frac{\mathrm{d}^3\mathbf{q}_a}{(2\pi)^3 2E_{\mathbf{q}_a}}\right\}\left\{\prod_{i=1}^{n}\frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}}\right\}(2\pi)^4\delta^{(4)}\Big(\sum_{i=1}^{n}P_i - Q - \sum_{a=1}^{m}Q_a\Big) \;. \tag{9.169}$$

Moreover, c is a statistical factor and $\mathcal{M}$ is the scattering amplitude.

Let now $f(x,\mathbf{q})$ be a particle distribution function; we assume its normalization to be so chosen that the total number density of particles at x is given by

$$n(x) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, f(x,\mathbf{q}) \;. \tag{9.170}$$

In particular, in thermal equilibrium, $f(x,\mathbf{q}) \equiv n_{\mathrm{F}}(E_{\mathbf{q}})$ (or $f(x,\mathbf{q}) \equiv n_{\mathrm{B}}(E_{\mathbf{q}})$ for bosons) is independent of the position x, and determined uniquely by the temperature (and by possible chemical potentials). At the same time, in vacuum, assuming a single plane wave regularized by a finite volume V, we would have

$$f(x,\mathbf{q}) = \frac{(2\pi)^3}{V}\delta^{(3)}(\mathbf{q}-\mathbf{q}_0) \;; \tag{9.171}$$

the logic is that then $n(x)$, as defined by Eq. (9.170), evaluates to $1/V$, while $f(x,\mathbf{q}_0) = 1$, where we made use of $\delta^{(3)}(\mathbf{0}) = V/(2\pi)^3$.

To convert Eq. (9.168) into a Boltzmann equation, we identify the decay rate Γ by $-\partial_t f/f$ for the plane wave case, and deform then the structure to be Lorentz-covariant:

$$E_{\mathbf{q}}\Gamma \Rightarrow -E_{\mathbf{q}}\frac{\partial f_N}{\partial t}\frac{1}{f_N} \Rightarrow -q^\alpha \frac{\partial f_N}{\partial x^\alpha}\frac{1}{f_N} \;. \tag{9.172}$$

We also modify the right-hand side of Eq. (9.168) by allowing for $1+m$ particles in the initial state, and adding Bose enhancement and Fermi blocking factors. Thereby

$$\begin{aligned} q^\alpha\frac{\partial f_N}{\partial x^\alpha} &= -\frac{c}{2}\sum_{m,n}\int \mathrm{d}\Phi_{N+m\to n} \\ &\times\;\; \big\{|\mathcal{M}|^2_{N+m\to n} f_N f_a\cdots f_m (1\pm f_i)\cdots(1\pm f_n) \\ &\quad -|\mathcal{M}|^2_{n\to N+m} f_i\cdots f_n (1\pm f_N)(1\pm f_a)\cdots(1\pm f_m)\big\} \;. \end{aligned} \tag{9.173}$$

Here $+$ applies for bosons and $-$ for fermions.

Let us compare this with Eq. (9.153). We may observe that Eq. (9.153) corresponds to the *gain terms* of Eq. (9.173) (i.e. the last row), since the right-handed neutrinos are (by assumption) non-thermal and escape: $f_N \equiv 0$. At the same time, to obtain a complete match, we should work out the scattering matrix elements, $|\mathcal{M}|^2$, and the statistical factors, c. The "strength" of the quantum field theoretic computation leading to Eq. (9.153) is that these automatically obtain their correct values; its weakness is the allowing for (perhaps partial) equilibration is not possible with Eq. (9.153), but can be achieved through the non-linear dependence of Eq. (9.173) on f_N.