9.4. Particle production rate

Let us consider a system where some particles interact strongly enough to remain in thermal equilibrium, while others interact so weakly that they cannot follow thermal equilibrium. We can imagine that particles of the latter type "escape" from the thermal system, either concretely (if the system is of finite size) or in an abstract sense (being still within the same volume as the thermal particles but not interacting with them). Familiar physical examples of such settings are (i) the "decoupling" of weakly interacting dark matter particles in *cosmology*; (ii) the production of electromagnetic "hard probes", such as photons and $\mu^-\mu^+$-pairs, from the QCD plasma generated in *heavy ion collision experiments*; (iii) the neutrino "emissivity" of neutron stars, constituting the most important process by which compact objects in *astrophysics* may cool down.

The purpose of this section is to develop the general formalism for addressing this phenomenon[44]. To keep the discussion concrete, we choose a particular example, however: the production of right-handed neutrinos within the model of Eq. (8.69),

$$\mathcal{L}_M = \frac{1}{2}\,\bar{\tilde{N}} i \,\partial\!\!\!/\, \tilde{N} - \frac{1}{2}\,M \bar{\tilde{N}}\tilde{N} - F_\alpha \bar{L}_\alpha \tilde{\phi}\, a_R \tilde{N} - F^*_\alpha \bar{\tilde{N}}\tilde{\phi}^\dagger a_L L_\alpha + \mathcal{L}_{\rm MSM} \;, \tag{9.86}$$

where we have added the Lagrangian $\mathcal{L}_{\rm MSM}$ of the Minimal Standard Model (MSM), describing the thermalized degrees of freedom. The goal for now is to derive a master equation relating the production rate of N's to a certain Green's function, evaluated already in Sec. 8.2. In the next section, we then combine all ingredients and show how the dark matter abundance can be evaluated in practice.

Let $\hat\rho$ be the density matrix of the full theory, incorporating all degrees of freedom, and $\hat H$ the corresponding full Hamiltonian operator. Then the equation for the density matrix is[45]

$$i\frac{{\rm d}\hat\rho(t)}{{\rm d}t} = [\hat H, \hat\rho(t)] \;. \tag{9.87}$$

We now split $\hat H$ in the form

$$\hat H = \hat H_{\rm MSM} + \hat H_{\rm N} + \hat H_{\rm int} \;, \tag{9.88}$$

where $\hat H_{\rm MSM}$ is the Hamiltonian of the MSM, $\hat H_{\rm N}$ is the free Hamiltonian of right-handed neutrinos, and $\hat H_{\rm int}$, which is proportional to the neutrino Yukawa couplings, contains the interactions between right-handed neutrinos and the particles of the MSM:

$$\hat H_{\rm int} = \int {\rm d}^3\mathbf{x} \left[F_\alpha \hat{\bar{L}}_\alpha \hat{\tilde{\phi}}\, a_R \hat N + F^*_\alpha \hat{\bar{N}} \hat{\tilde{\phi}}^\dagger a_L \hat L_\alpha \right] \;. \tag{9.89}$$

Here now $\hat N$ is a Majorana spinor field operator. To find the concentration of right-handed neutrinos, one has to solve Eq. (9.87) with some initial condition. We will *assume* that the initial concentration of right-handed neutrinos is zero, that is

$$\hat\rho(0) = \hat\rho_{\rm MSM} \otimes |0\rangle\langle 0| \;, \tag{9.90}$$

where $\hat\rho_{\rm MSM} = Z^{-1}_{\rm MSM} \exp(-\beta \hat H_{\rm MSM})$, $\beta \equiv 1/T$, is the equilibrium MSM density matrix at a temperature T, and $|0\rangle$ is the vacuum state for right-handed neutrinos.

Considering now $\hat H_0 = \hat H_{\rm MSM} + \hat H_{\rm N}$ as a "free" Hamiltonian, and $\hat H_{\rm int}$ as an interaction term, one can derive an equation for the density matrix in the interaction picture, $\hat\rho_{\rm I} \equiv \exp(i\hat H_0 t)\hat\rho\exp(-i\hat H_0 t)$,

[44] We follow the presentation in T. Asaka, M. Laine and M. Shaposhnikov, *On the hadronic contribution to sterile neutrino production,* JHEP 06 (2006) 053 [hep-ph/0605209].

[45]
$$i\frac{\rm d}{{\rm d}t}|\psi\rangle = \hat H|\psi\rangle \;, \quad -i\frac{\rm d}{{\rm d}t}\langle\psi| = \langle\psi|\hat H \;\Rightarrow\; i\frac{\rm d}{{\rm d}t}|\psi\rangle\langle\psi| = [\hat H, |\psi\rangle\langle\psi|] \;\Rightarrow\; i\frac{\rm d}{{\rm d}t}\hat\rho(t) = [\hat H, \hat\rho(t)] \;.$$

in the standard way:

$$
\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\rm I}(t) &= -\hat{H}_0\hat{\rho}_{\rm I} + e^{i\hat{H}_0 t}[\hat{H},\hat{\rho}(t)]e^{-i\hat{H}_0 t} + \hat{\rho}_{\rm I}\hat{H}_0 \\
&= -\hat{H}_0\hat{\rho}_{\rm I} + e^{i\hat{H}_0 t}[\hat{H}_0+\hat{H}_{\rm int},\hat{\rho}(t)]e^{-i\hat{H}_0 t} + \hat{\rho}_{\rm I}\hat{H}_0 \\
&= e^{i\hat{H}_0 t}[\hat{H}_{\rm int},\hat{\rho}(t)]e^{-i\hat{H}_0 t} \\
&= e^{i\hat{H}_0 t}\hat{H}_{\rm int}e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t} - e^{i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}\hat{H}_{\rm int}e^{-i\hat{H}_0 t} \\
&= [\hat{H}_{\rm I},\hat{\rho}_{\rm I}(t)]\ . \qquad (9.91)
\end{aligned}
$$

Here, as usual, $\hat{H}_{\rm I} = \exp(i\hat{H}_0 t)\hat{H}_{\rm int}\exp(-i\hat{H}_0 t)$ is the interaction Hamiltonian in the interaction picture. Now, perturbation theory with respect to $\hat{H}_{\rm I}$ can be used to compute the time evolution of $\hat{\rho}_{\rm I}$; the first two terms read

$$
\hat{\rho}_{\rm I}(t) = \hat{\rho}_0 - i\int_0^t \mathrm{d}t'\,[\hat{H}_{\rm I}(t'),\hat{\rho}_0] + (-i)^2\int_0^t \mathrm{d}t'\int_0^{t'}\mathrm{d}t''\,[\hat{H}_{\rm I}(t'),[\hat{H}_{\rm I}(t''),\hat{\rho}_0]] + \ldots\ , \qquad (9.92)
$$

where $\hat{\rho}_0 \equiv \hat{\rho}(0) = \hat{\rho}_{\rm I}(0)$. Note that perturbation theory as an expansion in $\hat{H}_{\rm I}$ may break down at a certain time $t \simeq t_{\rm eq}$ due to so-called secular terms. Physically, the reason is that after $t_{\rm eq}$ right-handed neutrinos enter thermal equilibrium and their concentration needs to be computed by other means. Here we assumed that $t \ll t_{\rm eq}$ and thus perturbation theory should work.

We are interested in the distribution function of the right-handed neutrinos. It is associated with the operator

$$
\frac{\mathrm{d}\hat{N}}{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}} \equiv \frac{1}{V}\sum_{s=\pm1}\hat{a}^\dagger_{\mathbf{q},s}\hat{a}_{\mathbf{q},s}\ , \qquad (9.93)
$$

where $\hat{a}^\dagger_{\mathbf{q},s}$ is the creation operator of a right-handed neutrino with momentum $\mathbf{q}$ and spin state s, normalised as

$$
\{\hat{a}_{\mathbf{p},s},\hat{a}^\dagger_{\mathbf{q},t}\} = \delta^{(3)}(\mathbf{p}-\mathbf{q})\delta_{st}\ , \qquad (9.94)
$$

and V is the volume of the system. Then the distribution function $\mathrm{d}N/\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}$ (number of right-handed neutrinos per $\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}$) is given by

$$
\frac{\mathrm{d}N(x,\mathbf{q})}{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}} \equiv \mathrm{Tr}\left[\frac{\mathrm{d}\hat{N}}{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{q}}\hat{\rho}_{\rm I}(t)\right]\ . \qquad (9.95)
$$

Inserting here Eq. (9.92), the first term leads to a time-independent result, and the second term does not contribute since $\hat{H}_{\rm I}$ is linear in $\hat{a}^\dagger_{\mathbf{q},s}$ and $\hat{a}_{\mathbf{q},s}$ (cf. Eqs. (9.89), (9.97), (9.98)). Thus, we get that to $\mathcal{O}(F^2)$ the rate of right-handed neutrino production reads

$$
\frac{\mathrm{d}N(x,\mathbf{q})}{\mathrm{d}^4x\,\mathrm{d}^3\mathbf{q}} = R(T,\mathbf{q}) \equiv -\frac{1}{V}\mathrm{Tr}\left\{\sum_{s=\pm1}\hat{a}^\dagger_{\mathbf{q},s}\hat{a}_{\mathbf{q},s}\int_0^t \mathrm{d}t'\,[\hat{H}_{\rm I}(t),[\hat{H}_{\rm I}(t'),\hat{\rho}_0]]\right\}\ . \qquad (9.96)
$$

The interaction Hamiltonian $\hat{H}_{\rm I}$ appearing in Eq. (9.96) has the form in Eq. (9.89), except that we now interpret the field operators as being in the interaction picture. Since $\hat{N}$ evolves with the free Hamiltonian $\hat{H}_{\rm N}$ in the interaction picture, it has the form of a free on-shell field operator, and can hence be written as

$$
\begin{aligned}
\hat{N}(x) &= \int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}}\sum_{s=\pm1}\left[\hat{a}_{\mathbf{p},s}u(\mathbf{p},s)e^{-iP\cdot x} + \hat{a}^\dagger_{\mathbf{p},s}v(\mathbf{p},s)e^{iP\cdot x}\right]\ , \qquad (9.97) \\
\hat{\bar{N}}(x) &= \int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}}\sum_{s=\pm1}\left[\hat{a}^\dagger_{\mathbf{p},s}\bar{u}(\mathbf{p},s)e^{iP\cdot x} + \hat{a}_{\mathbf{p},s}\bar{v}(\mathbf{p},s)e^{-iP\cdot x}\right]\ , \qquad (9.98)
\end{aligned}
$$

where we assumed the normalization in Eq. (9.94), and $p^0 \equiv E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + M^2}$, $P \equiv (p^0, \mathbf{p})$. The spinors u, v satisfy the completeness relations

$$\sum_s u(\mathbf{p},s)\bar{u}(\mathbf{p},s) = \not{P} + M \; , \sum_s v(\mathbf{p},s)\bar{v}(\mathbf{p},s) = \not{P} - M \; . \tag{9.99}$$

Inserting the free field operators into (the interaction picture version of) Eq. (9.89), we can rewrite $\hat{H}_{\rm I}$ as

$$\hat{H}_{\rm I} = \int {\rm d}^3\mathbf{x} \int \frac{{\rm d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{s=\pm 1} \Big[\hat{a}^\dagger_{\mathbf{p},s}\, \hat{J}_{\mathbf{p},s}(x)\, e^{iP\cdot x} + \hat{J}^\dagger_{\mathbf{p},s}(x)\, \hat{a}_{\mathbf{p},s}\, e^{-iP\cdot x} \Big] \; , \tag{9.100}$$

where, denoting

$$\begin{aligned} \hat{j}_\alpha(x) &\equiv \hat{\tilde{\phi}}^\dagger(x) \hat{L}_\alpha(x) \; , & (9.101) \\ \hat{\bar{j}}_\beta(x) &\equiv \hat{\bar{L}}_\beta(x) \hat{\tilde{\phi}}(x) \; , & (9.102) \end{aligned}$$

the operators can be written as

$$\begin{aligned} \hat{J}_{\mathbf{p},s}(x) &\equiv -F_\beta \hat{\bar{j}}_\beta(x) a_R v(\mathbf{p},s) + F^*_\alpha\, \bar{u}(\mathbf{p},s) a_L \hat{j}_\alpha(x) \; , & (9.103) \\ \hat{J}^\dagger_{\mathbf{p},s}(x) &\equiv -F^*_\alpha \bar{v}(\mathbf{p},s) a_L \hat{j}_\alpha(x) + F_\beta\, \hat{\bar{j}}_\beta(x) a_R u(\mathbf{p},s) \; . & (9.104) \end{aligned}$$

It remains to take the following steps:

(i) We insert Eq. (9.100) into Eq. (9.96) and remove the right-handed neutrino creation and annihilation operators, by making use of Eq. (9.94). We note first that

$$\begin{aligned} {\rm Tr}\Big\{ \hat{A}\; [\hat{B}, [\hat{C}, |0\rangle\, \langle 0|]] \Big\} &= {\rm Tr}\Big\{ \hat{A}\Big(\hat{B}\hat{C}|0\rangle\langle 0| - \hat{B}|0\rangle\langle 0|\hat{C} - \hat{C}|0\rangle\langle 0|\hat{B} + |0\rangle\langle 0|\hat{C}\hat{B} \Big) \Big\} \\ &= \langle 0| \Big\{ \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} - \hat{B}\hat{A}\hat{C} + \hat{C}\hat{B}\hat{A} \Big\} |0\rangle \; , \end{aligned} \tag{9.105}$$

where we denoted

$$\begin{aligned} \hat{A} &= \sum_{s=\pm 1} \hat{a}^\dagger_{\mathbf{q},s} \hat{a}_{\mathbf{q},s} \; , & (9.106) \\ \hat{B} &= \int {\rm d}^3\mathbf{x} \int \frac{{\rm d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}} \sum_{m=\pm 1} \Big[\hat{a}^\dagger_{\mathbf{p},m}\, \hat{J}_{\mathbf{p},m}(x)\, e^{iP\cdot x} + \hat{J}^\dagger_{\mathbf{p},m}(x)\, \hat{a}_{\mathbf{p},m}\, e^{-iP\cdot x} \Big] \; , & (9.107) \\ \hat{C} &= \int {\rm d}^3\mathbf{x}' \int \frac{{\rm d}^3\mathbf{r}}{\sqrt{(2\pi)^3 2r^0}} \sum_{n=\pm 1} \Big[\hat{a}^\dagger_{\mathbf{r},n}\, \hat{J}_{\mathbf{r},n}(x')\, e^{iR\cdot x'} + \hat{J}^\dagger_{\mathbf{r},n}(x')\, \hat{a}_{\mathbf{r},n}\, e^{-iR\cdot x'} \Big] \; . & (9.108) \end{aligned}$$

A non-zero trace only arises from structures of the type $\langle 0|\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle$, i.e. the second and third terms in Eq. (9.105). Thus, Eq. (9.96) becomes

$$\begin{aligned} R(T,\mathbf{q}) &= \frac{1}{V} \sum_{s,m,n=\pm 1} \int_0^t {\rm d}t' \int {\rm d}^3\mathbf{x} \int {\rm d}^3\mathbf{x}' \int \frac{{\rm d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2p^0}} \int \frac{{\rm d}^3\mathbf{r}}{\sqrt{(2\pi)^3 2r^0}} {\rm Tr}\Big\{ \hat{\rho}_{\rm MSM} \\ &\times \; \Big[\hat{J}^\dagger_{\mathbf{r},n}(x') \hat{J}_{\mathbf{p},m}(x) e^{iP\cdot x - iR\cdot x'} \langle 0|\hat{a}_{\mathbf{r},n}\hat{a}^\dagger_{\mathbf{q},s}\hat{a}_{\mathbf{q},s}\hat{a}^\dagger_{\mathbf{p},m}|0\rangle \\ &+ \; \hat{J}^\dagger_{\mathbf{p},m}(x) \hat{J}_{\mathbf{r},n}(x') e^{-iP\cdot x + iR\cdot x'} \langle 0|\hat{a}_{\mathbf{p},m}\hat{a}^\dagger_{\mathbf{q},s}\hat{a}_{\mathbf{q},s}\hat{a}^\dagger_{\mathbf{r},n}|0\rangle \Big] \Big\} \; . \end{aligned} \tag{9.109}$$

Both expectation values evaluate to

$$\langle 0|\hat{a}_{\mathbf{r},n}\hat{a}^\dagger_{\mathbf{q},s}\hat{a}_{\mathbf{q},s}\hat{a}^\dagger_{\mathbf{p},m}|0\rangle = \langle 0|\hat{a}_{\mathbf{p},m}\hat{a}^\dagger_{\mathbf{q},s}\hat{a}_{\mathbf{q},s}\hat{a}^\dagger_{\mathbf{r},n}|0\rangle = \delta_{ms}\delta_{ns}\delta^{(3)}(\mathbf{r}-\mathbf{q})\delta^{(3)}(\mathbf{p}-\mathbf{q}) \; . \tag{9.110}$$

Thereby

$$
\begin{aligned}
R(T,\mathbf{q}) &= \frac{1}{V}\frac{1}{(2\pi)^3 2E_{\mathbf{q}}}\sum_{s=\pm 1}\int_0^t \mathrm{d}t' \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}' \times \\
&\times \left\langle \hat{J}^\dagger_{\mathbf{q},s}(x')\hat{J}_{\mathbf{q},s}(x)e^{iQ\cdot(x-x')} + \hat{J}^\dagger_{\mathbf{q},s}(x)\hat{J}_{\mathbf{q},s}(x')e^{iQ\cdot(x'-x)}\right\rangle , \qquad (9.111)
\end{aligned}
$$

where from now on the expectation value refers to that with respect to $\hat{\rho}_{\mathrm{MSM}}$.

(ii) Inserting Eqs. (9.103), (9.104), we note that correlators of the type $\langle \hat{\bar{j}}_\beta(x')\hat{\bar{j}}_\alpha(x)\rangle$ and $\langle \hat{j}_\beta(x')\hat{j}_\alpha(x)\rangle$ must vanish, since lepton numbers are conserved within the MSM. The rate thus becomes

$$
\begin{aligned}
R(T,\mathbf{q}) &= \frac{1}{V}\frac{1}{(2\pi)^3 2E_{\mathbf{q}}}\sum_{s=\pm 1}\int_0^t \mathrm{d}t' \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\, F^*_\alpha F_\beta \times \\
&\times \Big\langle \Big[\bar{v}(\mathbf{q},s)a_L\hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x)a_R v(\mathbf{q},s) + \hat{\bar{j}}_\beta(x')a_R u(\mathbf{q},s)\bar{u}(\mathbf{q},s)a_L\hat{j}_\alpha(x)\Big]e^{iQ\cdot(x-x')} \\
&\quad +(x\leftrightarrow x')\Big\rangle . \qquad (9.112)
\end{aligned}
$$

(iii) The spinors u, v appear in a form where the standard completeness relations mentioned in Eq. (9.99) can be used. (In the first term, this requires writing

$$
\bar{v}(\mathbf{q},s)a_L\hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x)a_R v(\mathbf{q},s) = \mathrm{Tr}\left[v(\mathbf{q},s)\bar{v}(\mathbf{q},s)a_L\hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x)a_R\right] .) \qquad (9.113)
$$

The mass terms M that are induced this way get projected out by a_L, a_R. Therefore,

$$
\begin{aligned}
R(T,\mathbf{q}) &= \frac{1}{V}\frac{1}{(2\pi)^3 2E_{\mathbf{q}}}\sum_{s=\pm 1}\int_0^t \mathrm{d}t' \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\, F^*_\alpha F_\beta \times \\
&\times \Big\langle\Big\{\mathrm{Tr}\Big[\not{Q}\, a_L\hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x)a_R\Big] + \hat{\bar{j}}_\beta(x')a_R\not{Q}\, a_L\hat{j}_\alpha(x)\Big\}e^{iQ\cdot(x-x')} \\
&\quad +(x\leftrightarrow x')\Big\rangle . \qquad (9.114)
\end{aligned}
$$

If we for a moment generalize the notation such that α, β account for Lorentz indices as well as flavour indices, we can in fact rewrite this as

$$
\begin{aligned}
R(T,\mathbf{q}) &= \frac{1}{V}\frac{1}{(2\pi)^3 2E_{\mathbf{q}}}\sum_{s=\pm 1}\int_0^t \mathrm{d}t' \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\, F^*_\alpha F_\beta \times \\
&\times \Big\langle\Big\{(a_R\not{Q}\, a_L)_{\beta\alpha}\Big[\hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x) + \hat{\bar{j}}_\beta(x')\hat{j}_\alpha(x)\Big]\Big\}e^{iQ\cdot(x-x')} + (x\leftrightarrow x')\Big\rangle . \qquad (9.115)
\end{aligned}
$$

(iv) Recalling the notation in Eqs. (8.43), (8.44),

$$
\begin{aligned}
\Pi^>_{\alpha\beta}(Q) &\equiv \int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot(x-x')}\Big\langle \hat{j}_\alpha(x)\hat{\bar{j}}_\beta(x')\Big\rangle , \qquad (9.116) \\
\Pi^<_{\alpha\beta}(Q) &\equiv \int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\, e^{iQ\cdot(x-x')}\Big\langle -\hat{\bar{j}}_\beta(x')\hat{j}_\alpha(x)\Big\rangle , \qquad (9.117)
\end{aligned}
$$

and noting that translational invariance implies

$$
\Big\langle \hat{j}_\alpha(x)\hat{\bar{j}}_\beta(x')\Big\rangle = f(x-x') = f(-x'-(-x)) = \Big\langle \hat{j}_\alpha(-x')\hat{\bar{j}}_\beta(-x)\Big\rangle , \qquad (9.118)
$$

we can use Eq. (9.118) in the opposite direction and invert Eq. (9.116), to write

$$
\begin{aligned}
\Big\langle \hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x)\Big\rangle = \Big\langle \hat{j}_\alpha(-x)\hat{\bar{j}}_\beta(-x')\Big\rangle &= \int \frac{\mathrm{d}^4P}{(2\pi)^4}e^{iP\cdot(x-x')}\Pi^>_{\alpha\beta}(P) \\
&\overset{P\to -P}{=} \int \frac{\mathrm{d}^4P}{(2\pi)^4}e^{-iP\cdot(x-x')}\Pi^>_{\alpha\beta}(-P) . \qquad (9.119)
\end{aligned}
$$

Therefore, the two-point correlator in Eq. (9.115) can be written as

$$\langle \hat{j}_\alpha(x')\hat{\bar{j}}_\beta(x) + \hat{\bar{j}}_\beta(x')\hat{j}_\alpha(x)\rangle = \int \frac{\mathrm{d}^4 P}{(2\pi)^4} e^{-iP\cdot(x-x')}\Big[\Pi^{>}_{\alpha\beta}(-P) - \Pi^{<}_{\alpha\beta}(P)\Big] . \tag{9.120}$$

(v) It remains to carry out the integrals over the space and time coordinates. Taking the limit $t \to \infty$ and summing both terms in Eq. (9.115) together, yields

$$\begin{aligned}
& \lim_{t\to\infty} \int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{x}' \int_0^t \mathrm{d}t' \Big[e^{i(Q-P)\cdot(x-x')} + e^{i(P-Q)\cdot(x-x')}\Big] \\
= \;& V(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \lim_{t\to\infty}\int_0^t \mathrm{d}t' \Big[e^{i(q^0-p^0)(t-t')} + e^{i(p^0-q^0)(t-t')}\Big] \\
= \;& V(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \lim_{t\to\infty}\bigg\{\int_{-t}^0 \mathrm{d}t'' \Big[e^{i(p^0-q^0)t''} + e^{-i(p^0-q^0)t''}\Big]\bigg\}_{t''\equiv t'-t} \\
= \;& V(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \lim_{t\to\infty}\bigg\{\int_{-t}^0 \mathrm{d}t''\, e^{i(p^0-q^0)t''} + \int_0^t \mathrm{d}t'''\, e^{i(p^0-q^0)t'''}\bigg\}_{t'''\equiv -t''} \\
= \;& V(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \int_{-\infty}^{\infty} \mathrm{d}\tilde{t}\, e^{i(p^0-q^0)\tilde{t}} = V(2\pi)^4\delta^{(4)}(P-Q) ,
\end{aligned} \tag{9.121}$$

which allows to cancel $1/V$ from Eq. (9.115) and remove P-integration from Eq. (9.120).

As a result of all these steps, we obtain ($q^0 \equiv E_{\mathbf{q}}$)

$$R(T,\mathbf{q}) = \frac{1}{(2\pi)^3 2q^0} F^*_\alpha F_\beta \mathrm{Tr}\,\Big\{\not{Q}\, a_L \Big[\Pi^{>}_{\alpha\beta}(-Q) - \Pi^{<}_{\alpha\beta}(Q)\Big] a_R\Big\} , \tag{9.122}$$

where we have returned to the convention that α, β label generations, and have expressed the Dirac part through a trace. Making use of the fact that $1 - n_{\mathrm{F}}(-q^0) = n_{\mathrm{F}}(q^0)$, Eq. (8.53) yields

$$\begin{aligned}
\Pi^{>}_{\alpha\beta}(-Q) &= 2[1-n_{\mathrm{F}}(-q^0)]\rho_{\alpha\beta}(-Q) = 2n_{\mathrm{F}}(q^0)\rho_{\alpha\beta}(-Q) , && (9.123)\\
\Pi^{<}_{\alpha\beta}(Q) &= -2n_{\mathrm{F}}(q^0)\rho_{\alpha\beta}(Q) . && (9.124)
\end{aligned}$$

Observing furthermore that lepton generation conservation within the MSM restricts the indices α, β to be equal, we finally obtain the master relation

$$R(T,\mathbf{q}) = \frac{2n_{\mathrm{F}}(q^0)}{(2\pi)^3 2q^0} \sum_{\alpha=1}^{3} |F_\alpha|^2 \mathrm{Tr}\,\Big\{\not{Q}\, a_L \Big[\rho_{\alpha\alpha}(-Q) + \rho_{\alpha\alpha}(Q)\Big] a_R\Big\} . \tag{9.125}$$

We stress again that this relation is valid only provided that the number density of right-handed neutrinos created is much smaller than their equilibrium concentration.

In summary, we have obtained a relation of the particle production rate, Eq. (9.96), to a finite-temperature spectral function, computed already in Eq. (8.88).