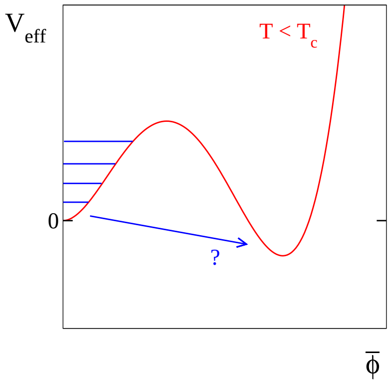
9.2. Bubble nucleation rate

It was already mentioned in the previous section that if a first order transition takes place, its dynamics is non-trivial, because the latent heat releases in the transition needs to be dissipated away. The basic mechanism for this is *bubble nucleation and growth*: the transition does not take place exactly at the critical temperature, T_c, but the system first supercools to some nucleation temperature, T_n. At this point bubbles of the stable phase form, and start to grow; the latent heat is transported away in a hydrodynamic shock wave which precedes the expanding bubble.

The purpose of this section is to determine the probability of bubble nucleation, per unit time and volume, at a given temperature $T < T_c$. Combined with the cosmological evolution equation for the temperature T as a function of the time t, this would in principle allow us to estimate T_n. We will not get into explicit estimates, though, but rather try to illustrate some essential aspects of the general formalism, given that it is analogous to several other "rate" computations in quantum field theory, such as the determination of the rate of baryon number violation in the standard electroweak theory. In terms of the effective potential, the general setting can be illustrated as follows:



For simplicity we consider the situation in the following that a bump already exists in the tree-level potential $V(\bar\phi)$; for radiatively generated transitions, in which a bump only appears in $V_{\rm eff}(\bar\phi)$, some degrees of freedom need to be integrated out for this to be the case.

The starting point now is to properly *define* what is meant with the nucleation rate. It turns out that this task is rather non-trivial; in fact, it is not clear whether a completely general non-perturbative definition can be given at all. Nevertheless, for practical purposes, the Langer formalism[37] appears sufficient.

The general idea is the following. Consider first a system at zero temperature. Suppose we use *boundary conditions* at spatial infinity, $\lim_{|\mathbf{x}|\to\infty}\phi(\mathbf{x}) = 0$, in order to define metastable energy eigenstates. We could imagine that their time evolution looks like

$$|\phi(t)\rangle = e^{-iEt}|\phi(0)\rangle = e^{-i[\mathrm{Re}(E)+i\,\mathrm{Im}(E)]t}|\phi(0)\rangle \tag{9.36}$$

$$\Rightarrow \quad \langle\phi(t)|\phi(t)\rangle = e^{2\,\mathrm{Im}(E)\,t}\langle\phi(0)|\phi(0)\rangle\ . \tag{9.37}$$

Thereby we could say that such a metastable state possesses a decay rate, $\Gamma(E)$, given by

$$\Gamma(E) = -2\,\mathrm{Im}(E)\ . \tag{9.38}$$

Moving then to a thermal ensemble, we could analogously expect that

$$\Gamma = -2\,\mathrm{Im}\,\Omega\ , \tag{9.39}$$

[37] J.S. Langer, *Theory of the condensation point*, Ann. Phys. 41 (1967) 108; *Statistical theory of the decay of metastable states*, Ann. Phys. 54 (1969) 258.

where Ω is the grand canonical free energy. It should be stressed, though, that this equation is to be taken kind of as a definition for the moment: it would almost be a miracle if a real-time observable, the nucleation rate, could really be determined exactly from a static Euclidean observable, the free energy.

In any case, we can now pose the question whether Ω could indeed develop an imaginary part? It turns out that it can, as can be seen with the following argument[38]. Consider the path integral expression for the partition function,

$$\Omega = -T \ln \biggl\{ \int_{\text{b.c.}} \mathcal{D}\phi \exp\Bigl(-S_E[\phi]\Bigr) \biggr\} , \tag{9.40}$$

where "b.c." refers to the usual periodic boundary conditions. Let us assume that we can find *two* different saddle points, each satisfying

$$\frac{\delta S_E}{\delta \phi}\bigg|_{\phi=\hat{\phi}} = 0 , \quad \hat{\phi}(0,\mathbf{x}) = \hat{\phi}(\beta,\mathbf{x}) , \quad \lim_{|\mathbf{x}|\to\infty} \hat{\phi}(\tau,\mathbf{x}) = 0 . \tag{9.41}$$

One solution is a trivial one, $\phi \equiv 0$, and the key assumption is that there is also a non-trivial solution which we denote by $\hat{\phi}(\tau,\mathbf{x})$.

Let us then consider fluctuations around the saddle point. Suppose that the fluctuation operator around $\hat{\phi}$ has one *negative eigenmode*:

$$\frac{\delta^2 S_E}{\delta \phi^2}\bigg|_{\phi=\hat{\phi}} f_-(\tau,\mathbf{x}) = -\lambda_-^2 f_-(\tau,\mathbf{x}) . \tag{9.42}$$

(For the non-negative modes, we define the eigenvalues through

$$\frac{\delta^2 S_E}{\delta \phi^2}\bigg|_{\phi=\hat{\phi}} f_n(\tau,\mathbf{x}) = \lambda_n^2 f_n(\tau,\mathbf{x}) .) \tag{9.43}$$

Then, with $Z_0 \equiv Z[\phi = 0]$, the grand canonical potential gets contributions from both saddle points:

$$\Omega \;\sim\; -T \ln \biggl\{ Z_0 \;+ e^{-S_E[\hat{\phi}]} \int \frac{\mathrm{d}f_-}{\sqrt{2\pi}}\, e^{\frac{1}{2}\lambda_-^2 f_-^2} \int \prod_{n\ge 0} \frac{\mathrm{d}f_n}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\lambda_n^2 f_n^2} \biggr\} . \tag{9.44}$$

The negative eigenmode requires a careful analysis, but essentially

$$\int \frac{\mathrm{d}f_-}{\sqrt{2\pi}}\, e^{\frac{1}{2}\lambda_-^2 f_-^2} \sim \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{-\lambda_-^2}} \sim i \sqrt{\frac{1}{\lambda_-^2}} , \tag{9.45}$$

so that there indeed is an imaginary part. Assuming that the partition function around the regular saddle point is much larger in absolute magnitude than that around the non-trivial one, then leads to

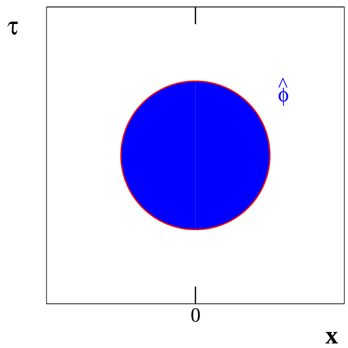
$$\Gamma \sim \frac{T}{Z_0} \exp\Bigl\{-S_E[\hat{\phi}]\Bigr\} \Bigl|\det\Bigl(\delta^2 S_E[\hat{\phi}]/\delta\phi^2\Bigr)\Bigr|^{-\frac{1}{2}} . \tag{9.46}$$

Such formulae are generically referred to as the *semiclassical approximation.* Somewhat more precise versions of this formula will be given in Eqs. (9.61), (9.64) below.

The non-trivial saddle point is generally referred to as an *instanton.* By definition, an instanton is a solution of the imaginary-time classical equations of motion, but it describes the rate of real-time transitions.

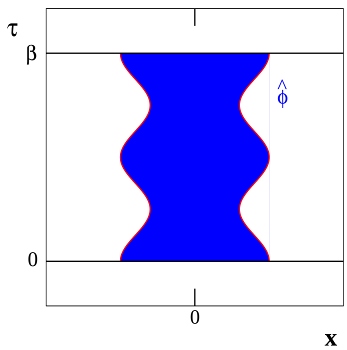
[38] S.R. Coleman, *The Fate of the False Vacuum. 1. Semiclassical Theory,* Phys. Rev. D 15 (1977) 2929 [Erratum-ibid. D 16 (1977) 1248].

Of course, the instanton needs to respect the boundary conditions in Eq. (9.41). Depending on the geometric shape of the instanton solution within these constraints, we can give different physical interpretations to the kind of "tunnelling" that the instanton describes. In the simplest case, when the temperature is very small (β is very large), the Euclidean time direction is identical with the space directions, and we can expect that the solution has 4d rotational symmetry:



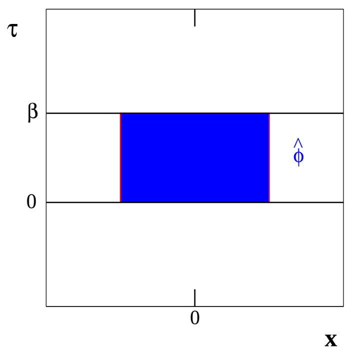
Such a solution is said to describe "quantum tunnelling". Indeed, had we kept $\hbar \neq 1$, Eq. (9.46) would have had the exponential $\exp(-S_E[\hat{\phi}]/\hbar)$.

On the other hand, if the temperature increases and β decreases, the four-volume becomes "squeezed", and this affects the form of the solution[39]. The solution can be depicted as follows:



Then we can say that "quantum tunnelling" and "thermal tunnelling" both play a role.

Eventually, however, the box becomes very squeezed, and we expect that the solution only respects 3d rotational symmetry:



[39] A.D. Linde, *Fate of the False Vacuum at Finite Temperature: Theory and Applications,* Phys. Lett. B 100 (1981) 37.

In this situation, like in dimensional reduction, we can factorize the integration over the τ-coordinate, and the instanton action becomes

$$\frac{1}{\hbar}S_E[\hat{\phi}] = \frac{1}{\hbar}\beta\hbar \int \mathrm{d}^3\mathbf{x}\, \mathcal{L}_E \equiv \beta\, S_{3\mathrm{d}}[\hat{\phi}]\ . \tag{9.47}$$

We say that then the transition takes place through "classical thermal tunnelling"

In typical cases, the exponential factor, be it the thermal one, $\beta S_{3\mathrm{d}}[\hat{\phi}]$, or the quantum mechanical one, $S_E[\hat{\phi}]/\hbar$, is very large. Thereby the exponential is very small, and just how small it is, is determined predominantly by the instanton action, rather than the fluctuation determinant which does not have any exponential factors, and is therefore "of order unity". Hence we can say that the instanton solution and its Euclidean action $S_E[\hat{\phi}]$ play the dominant role in determining the nucleation rate.

At the same time, from a theoretical point of view, it can be said that the real "art" in solving the problem, is the computation of the fluctuation determinant around the saddle point solution[40]. In fact, the eigenmodes of the fluctuation operator can be classified into:

(1) one negative mode;

(2) a number of zero-modes;

(3) infinitely many positive modes.

We have already addressed the negative mode, which is the one responsible for the imaginary part, so let us now look at the zero modes.

The existence and multiplicity of zero modes can be deduced from the classical equations of motion and from the expression of the fluctuation operator. Indeed, assuming

$$S_E \;=\; \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + V(\phi)\right]\ , \tag{9.48}$$

the classical equations of motion read

$$\delta S_E[\hat{\phi}]/\delta\phi = 0 \quad \Leftrightarrow \quad -\partial_\mu^2\hat{\phi} + V'(\hat{\phi}) = 0\ . \tag{9.49}$$

The fluctuation operator is given by

$$\delta^2 S_E[\hat{\phi}]/\delta\phi^2 \;=\; -\partial_\mu^2 + V''(\hat{\phi})\ , \tag{9.50}$$

and differentiating Eq. (9.49) by ∂_ν yields the equation

$$\left[-\partial_\mu^2 + V''(\hat{\phi})\right]\partial_\nu\hat{\phi} = 0\ . \tag{9.51}$$

Therefore, $\partial_\nu\hat{\phi}$ is a zero mode. Note that a zero-mode exists (i.e. is non-trivial) only if the solution $\hat{\phi}$ depends on the coordinate x^ν; thus, the trivial saddle point $\phi = 0$ does *not* lead to any zero modes.

A non-trivial issue related to the zero modes is their proper normalization. In fact, the integrals over the zero modes are only defined in a finite volume; they are proportional to the volume, $V = L^3$, corresponding to translations of the "critical bubble". A proper normalisation shows that

$$\int \frac{\mathrm{d}f_0}{\sqrt{2\pi}} = \left(\frac{\hat{S}_E}{2\pi}\right)^{\frac{1}{2}} L \tag{9.52}$$

[40] C.G. . Callan and S.R. Coleman, *The Fate of the False Vacuum. 2. First Quantum Corrections,* Phys. Rev. D 16 (1977) 1762.

for $\partial_1\hat{\phi}$, $\partial_2\hat{\phi}$, $\partial_3\hat{\phi}$, and $L \to \beta$ for $\partial_0\hat{\phi}$. This can be shown as follows.

Let us for simplicity consider a one-dimensional case (extent L). We normalize the eigenmodes of the fluctation operator, defined as in Eq. (9.42), through

$$\int_0^L \mathrm{d}x\, f_m f_n = \delta_{mn} \;, \tag{9.53}$$

and write a generic deviation from the saddle point solution as

$$\delta\phi = \phi - \hat{\phi} = \sum_n \delta\phi_n \equiv \sum_n c_n f_n \;, \tag{9.54}$$

where c_n are coefficients (which we assume, for simplicity, to be real). We now need to be more precise about the integration measure in Eq. (9.44); in fact it is sensible to define

$$\int \mathcal{D}\phi \equiv \prod_n \int \frac{\mathrm{d}c_n}{\sqrt{2\pi}} \;, \tag{9.55}$$

the point being that then the fluctuation determinant obtains a simple expression,

$$\begin{aligned}
\prod_n \biggl\{ \int_{-\infty}^{\infty} \frac{\mathrm{d}c_n}{\sqrt{2\pi}} \exp\biggl(- \int_0^L \mathrm{d}x\, \frac{1}{2}\, \delta\phi_n \frac{\delta^2 S_E[\hat{\phi}]}{\delta\phi^2} \delta\phi_n \biggr) \biggr\} \\
&= \prod_n \biggl\{ \int_{-\infty}^{\infty} \frac{\mathrm{d}c_n}{\sqrt{2\pi}} \exp\biggl(- \int_0^L \mathrm{d}x\, \frac{1}{2}\, \lambda_n c_n^2 \biggr) \biggr\} \\
&= \prod_n \frac{1}{\sqrt{\lambda_n}} \\
&= \frac{1}{\sqrt{\det\{\delta^2 S_E[\hat{\phi}]/\delta\phi^2\}}} \;.
\end{aligned} \tag{9.56}$$

Now, we return to the question of how to deal with the zero modes. The first task is to normalize them according to Eq. (9.53). We note that the classical equation of motion, Eq. (9.49), implies (by multiplying with $\partial_x\hat{\phi}$ and fixing the integration constant at infinity) a “virial theorem”, $\frac{1}{2}\,(\partial_x\hat{\phi})^2 = V(\hat{\phi})$. We then note that

$$\int_0^L \mathrm{d}x\, (\partial_x\hat{\phi})^2 = \int_0^L \mathrm{d}x\, \Bigl[\frac{1}{2}\,(\partial_x\hat{\phi})^2 + V(\hat{\phi}) \Bigr] = S_E[\hat{\phi}] \equiv \hat{S}_E \;. \tag{9.57}$$

In other words, the properly normalized zero-mode reads

$$f_0 = \frac{1}{\sqrt{\hat{S}_E}} \partial_x\hat{\phi} \;. \tag{9.58}$$

As the last step, we note that

$$c_0\, f_0(x) = \frac{1}{\sqrt{\hat{S}_E}}\, c_0\, \partial_x\hat{\phi}(x) \simeq \hat{\phi}\Bigl(x + \frac{1}{\sqrt{\hat{S}_E}} c_0 \Bigr) - \hat{\phi}(x) \;. \tag{9.59}$$

Since the box is of size L, this means that we should restrict $c_0 \in (0, \sqrt{\hat{S}_E}L)$, in order not to double-count. Correspondingly,

$$\int \frac{\mathrm{d}c_0}{\sqrt{2\pi}} = \biggl(\frac{\hat{S}_E}{2\pi} \biggr)^{\frac{1}{2}} L \;, \tag{9.60}$$

in accordance with Eq. (9.52).

We are now ready to put everything together. A more careful analysis[40] shows that the factor 2 in Eq. (9.39) cancels against a factor 1/2 which we missed in Eq. (9.45). Thereby Eq. (9.46) is correct expect for the treatment of the zero modes. Rectifying this point according to Eq. (9.52), and expressing also $\mathcal{Z}_0$ in the Gaussian approximation, we arrive at

$$\frac{\Gamma}{V} \simeq \left(\frac{\hat{S}_E}{2\pi}\right)^{\frac{4}{2}} \left|\frac{\det'[-\partial^2 + V''(\hat{\phi})]}{\det[-\partial^2 + V''(0)]}\right|^{-\frac{1}{2}} e^{-\hat{S}_E} , \tag{9.61}$$

where $\det'$ means that zero-modes have been omitted (but the negative mode is kept).

On the other hand, in the classical limit, $\partial_\tau \hat{\phi} = 0$. Thereby there are only three zero-modes, and

$$-2\,\mathrm{Im}\,\Omega \simeq TV \left(\frac{\hat{S}_{3\mathrm{d}}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det'[-\nabla^2 + V''(\hat{\phi})]}{\det[-\nabla^2 + V''(0)]}\right|^{-\frac{1}{2}} e^{-\beta \hat{S}_{3\mathrm{d}}} . \tag{9.62}$$

Furthermore, it turns out that the definition $\Gamma = -2\,\mathrm{Im}\,\Omega$ in Eq. (9.39) needs to be corrected into[41]

$$\Gamma \simeq -\frac{\beta\lambda_-}{\pi}\,\mathrm{Im}\,\Omega . \tag{9.63}$$

The result for the nucleation rate is thus

$$\left.\frac{\Gamma}{V}\right|_{\hbar=0} \simeq \left(\frac{\lambda_-}{2\pi}\right) \left(\frac{\hat{S}_{3\mathrm{d}}}{2\pi T}\right)^{\frac{3}{2}} \left|\frac{\det'[-\nabla^2 + V''(\hat{\phi})]}{\det[-\nabla^2 + V''(0)]}\right|^{-\frac{1}{2}} e^{-\frac{\hat{S}_{3\mathrm{d}}}{T}} . \tag{9.64}$$

Comparing Eqs. (9.39), (9.63), we may expect Eq. (9.64) to be more accurate than Eq. (9.61) for

$$T \gtrsim \frac{\lambda_-}{2\pi} . \tag{9.65}$$

It should be mentioned, though, the it is not clear whether the simplistic approach based on the negative eigenmode λ_- really gives the correct answer[42]; rather, we should understand the analysis in the sense that a rate exists, and the formulae as giving its rough order of magnitude.

Let us end by commenting on the analogous computation of the rate of baryon number violation. In that case, the two vacua are actually degenerate; and the role of the field $\bar{\phi}$ is played by the Chern-Simons number, which is a suitable coordinate for classifying topologically distinct vacua. However, the formalism itself is identical: in particular at low temperatures there is a saddle point solution with 4d symmetry, which is just the usual instanton, while at high temperatures the saddle point solution has 3d symmetry, and is referred to as *sphaleron.* Again there are also zero modes, which have to be treated separately[43], as well as infinitely many positive modes.

[41] I. Affleck, *Quantum Statistical Metastability,* Phys. Rev. Lett. 46 (1981) 388.

[42] P. Arnold, D. Son and L.G. Yaffe, *The hot baryon violation rate is $\mathcal{O}(\alpha_w^5 T^4)$,* Phys. Rev. D 55 (1997) 6264.

[43] P. Arnold and L.D. McLerran, *Sphalerons, Small Fluctuations and Baryon Number Violation in Electroweak Theory,* Phys. Rev. D 36 (1987) 581.

9.3. Exercise 13

In scalar field theory, the determination of the saddle point solution (i.e. the "critical bubble") and its Euclidean action becomes solvable if we assume (i) the classical limit of high temperatures and (ii) that the minima are almost degenerate. Show that then

$$\hat{S}_{3d} = \frac{16\pi}{3} \frac{\sigma^3}{(\Delta p)^2} \,, \tag{9.66}$$

where

$$\sigma \equiv \int_0^{\bar{\phi}_{\rm broken}} {\rm d}\bar{\phi} \sqrt{2V(\bar{\phi})} \tag{9.67}$$

is the *surface tension*, and

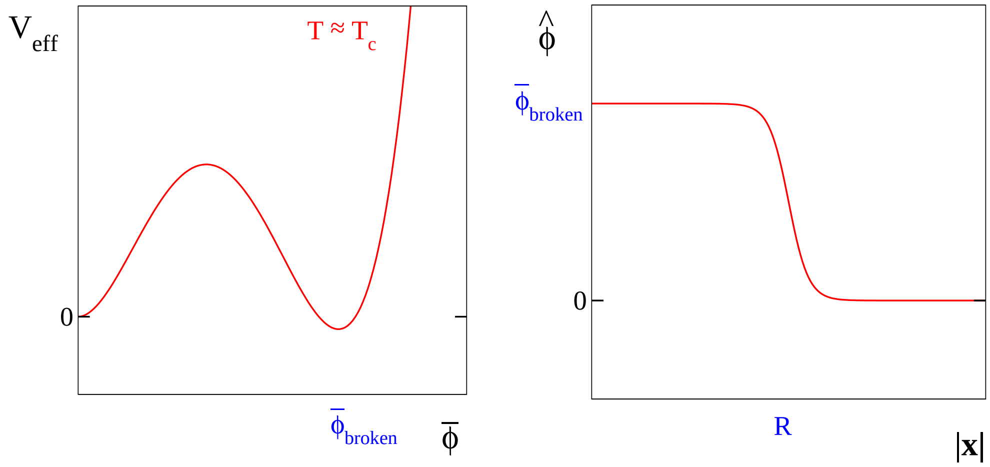
$$\Delta p \equiv V(0) - V(\bar{\phi}_{\rm broken}) \tag{9.68}$$

is the *pressure difference* in favour of the broken phase. Note that in this limit the configuration $\hat{\phi}$ is a *thin-wall bubble.*

Eq. (9.66) implies that $\hat{S}_{3d} \to \infty$ for $\Delta p \to 0$. Therefore nucleation can only take place after some supercooling, whereby $\Delta p > 0$ and $\hat{S}_{3d}$ becomes finite.

Solution to Exercise 13

The situation can be illustrated as follows:



In the classical limit, where there is no dependence on τ, the equation of motion becomes three-dimensional,

$$-\nabla^2 \hat{\phi} + V'(\hat{\phi}) = 0 \,. \tag{9.69}$$

Assuming now *spherical symmetry*, this can be written as

$$\frac{{\rm d}^2\hat{\phi}}{{\rm d}r^2} + \frac{2}{r}\frac{{\rm d}\hat{\phi}}{{\rm d}r} = V'(\hat{\phi}) \,. \tag{9.70}$$

The boundary conditions in Eq. (9.41) can be rephrased as

$$\begin{cases} \hat{\phi}(\infty) = 0 \\ \frac{{\rm d}\hat{\phi}(0)}{{\rm d}r} = 0 \end{cases} \,, \tag{9.71}$$

and the action reads

$$\hat{S}_{3\mathrm{d}} = 4\pi \int_0^\infty \mathrm{d}r\, r^2 \left\{ \frac{1}{2} \left(\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \right)^2 + V(\hat{\phi}) \right\} . \tag{9.72}$$

Before proceeding, it is useful to note that Eqs. (9.70), (9.71) have a mechanical analogue. Indeed, rewriting $r \to t$, $V \to -U$, $\hat{\phi} \to x$, they correspond to a classical "particle in a valley"-problem with friction. The particle starts at $t = 0$ from $x > 0$, near the top of the hill in the potential U; and rolls then towards the other top of the hill at the origin. The starting point has to be slightly higher than the end point, because the second term in Eq. (9.70) acts as friction, and eats up part of the energy. Therefore the broken minimum in V has to be *lower* than the symmetric one in order for a solution to exist.

Proceeding now with the solution, we introduce the following ansatz. Suppose that at $r < R$, the field is constant and has a value close to that in the broken minimum:

$$\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \simeq 0 \; , \quad V(\hat{\phi}) \simeq V(\bar{\phi}_{\mathrm{broken}}) \; . \tag{9.73}$$

The contribution to the action from this region is then

$$\hat{S}_{3\mathrm{d}} \simeq \frac{4}{3} \pi R^3 V(\bar{\phi}_{\mathrm{broken}}) \; . \tag{9.74}$$

For $r > R$, we assume a similar situation, but now the field is close to the origin:

$$\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \simeq 0 \; , \quad V(\hat{\phi}) \simeq V(0) \equiv 0 \; . \tag{9.75}$$

This region does therefore not contribute to the action. Finally, let us inspect the region at $r \simeq R$. If R is very large, the term $2\hat{\phi}'/R$ in Eq. (9.70) is very small, and can be neglected. Thereby

$$\frac{\mathrm{d}^2\hat{\phi}}{\mathrm{d}r^2} \simeq V'(\hat{\phi}) \; , \tag{9.76}$$

which can through multiplication with $\hat{\phi}'(r)$ be integrated into

$$\frac{1}{2} \left(\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \right)^2 \simeq V(\hat{\phi}) \; . \tag{9.77}$$

The contribution to the action becomes

$$\begin{aligned} \hat{S}_{3\mathrm{d}} &\simeq 4\pi R^2 \int_{R-\delta}^{R+\delta} \mathrm{d}r \left(\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \right)^2 \\ &\simeq 4\pi R^2 \int_0^{\bar{\phi}_{\mathrm{broken}}} \mathrm{d}\hat{\phi}\, \frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \\ &\simeq 4\pi R^2 \int_0^{\bar{\phi}_{\mathrm{broken}}} \mathrm{d}\hat{\phi} \sqrt{2V(\hat{\phi})} \; . \end{aligned} \tag{9.78}$$

The quantity

$$\sigma \equiv \int_{R-\delta}^{R+\delta} \mathrm{d}r \left\{ \frac{1}{2} \left(\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}r} \right)^2 + V(\hat{\phi}) \right\} \simeq \int_0^{\bar{\phi}_{\mathrm{broken}}} \mathrm{d}\hat{\phi} \sqrt{2V(\hat{\phi})} \tag{9.79}$$

represents the energy density of a planar surface, i.e. a *surface tension*.

Summing up the contributions to the action, we get

$$\hat{S}_{3\mathrm{d}}(R) \simeq 4\pi R^2 \sigma - \frac{4}{3} \pi R^3 \Delta p \; , \tag{9.80}$$

where $\Delta p > 0$ was defined according to Eq. (9.68). The so far free parameter R can now be determined by extremizing,

$$\delta_R \hat{S}_{3\mathrm{d}} = 0 \ , \tag{9.81}$$

leading to the radius $R = 2\sigma/\Delta p$. Substituting back to Eq. (9.80), the saddle point action becomes

$$\hat{S}_{3\mathrm{d}} = 4\pi\sigma \frac{4\sigma^2}{(\Delta p)^2} - \frac{4}{3}\pi \frac{8\sigma^3}{(\Delta p)^2} = \frac{16\pi}{3} \frac{\sigma^3}{(\Delta p)^2} \ . \tag{9.82}$$

Finally, if we are very close to T_c, Δp here can be related to basic characteristics of the first order transition. Indeed, the energy density is

$$e = Ts - p \tag{9.83}$$

where the entropy density reads $s = \mathrm{d}p/\mathrm{d}T$. Across the transition, the pressure is continuous but the energy density has a discontinuity, called the latent heat:

$$L \equiv -\Delta e = -T_\mathrm{c}\Delta s = -T_\mathrm{c}\frac{\mathrm{d}}{\mathrm{d}T}\Delta p \ . \tag{9.84}$$

Therefore

$$\Delta p(T) \approx \Delta p(T_\mathrm{c}) + \frac{\mathrm{d}\Delta p}{\mathrm{d}T}(T - T_\mathrm{c}) = L\Big(1 - \frac{T}{T_\mathrm{c}}\Big) \ . \tag{9.85}$$

At the same time, the surface tension remains finite at the transition point. Thereby the nucleation action in Eq. (9.82) diverges rapidly as $T \to T_\mathrm{c}^-$.