9. Applications

9.1. Thermal phase transitions

As a first application of the general formalism developed, we consider the existence of finite-temperature phase transitions in models of particle physics. Prime examples are the "deconfinement" transition taking place in QCD, and the "electroweak symmetry restoring" transition taking place in the electroweak theory[32], both of which could have significance for Early Universe cosmology. For simplicity, though, the practical analysis will be carried out within the scalar field theory discussed in Sec. 3.

In general, a phase transition can be defined to be a line in the (T, μ)-plane across which the grand canonical free energy density $f(T, \mu)$ is non-analytic. In particular, if $\partial f / \partial T$ or $\partial f / \partial \mu$ is discontinuous, we speak of a *first order transition*. The energy density

$$e = \frac{1}{V\mathcal{Z}}\text{Tr}\left[\hat{H}e^{-\beta(\hat{H}-\mu\hat{Q})}\right] = \frac{T^2}{V}\frac{\partial}{\partial T}\left(\ln\mathcal{Z}\right)_{\frac{\mu}{T}} = f - T\left(\frac{\partial f}{\partial T}\right)_{\frac{\mu}{T}} \tag{9.1}$$

is then discontinuous. This means that a closed system can proceed through the transition only if there is some mechanism for energy transfer and dissipation; thus, first order transitions always possess non-trivial dynamics.

It is often possible to associate an *order parameter* with a phase transition. In a strict sense, the order parameter should be an elementary or composite field, the expectation value of which vanishes in one phase and is non-zero in another. In a generalized sense, we may refer to an order parameter even if it would not vanish in either phase, provided that (in first order transitions) it jumps across the phase boundary. A particularly simple situation is if this role is taken by some elementary field; in the following we consider the case where a real scalar field, ϕ, plays the role of an order parameter. More realistically, ϕ could be for instance a neutral component of the Higgs doublet (in some gauge) in, say, the MSSM.

The Euclidean Lagrangian of the system then reads

$$\mathcal{L}_E = \frac{1}{2}\left(\partial_\tau\phi\right)^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + V(\phi) . \tag{9.2}$$

We take the potential to be of the form $V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$, like in the Standard Model, so that ϕ has a non-zero expectation value at zero temperature:
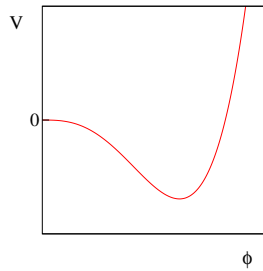
Let us now evaluate the partition function of this system with the method of the *effective potential*, $V_{\text{eff}}(\bar{\phi})$, introduced in Sec. 7.2. In other words, we again put the system in a finite volume V, and denote by $\bar{\phi}$ the *condensate*, i.e. the mode with $\mathbf{p} = 0, \omega_{\text{b}} = 0$. We then write

$$\mathcal{Z}(V, T, \mu) = \int_{-\infty}^{\infty}\mathrm{d}\bar{\phi}\int_{\tilde{P}\neq 0}\mathcal{D}\phi' \; e^{-S_E[\phi=\bar{\phi}+\phi']} \tag{9.3}$$

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[32]We use here the standard characterizations of these transitions, even though in a strict sense neither concept is appropriate.

$$\equiv \int_{-\infty}^{\infty} \mathrm{d}\bar{\phi} \, \exp\left\{-\frac{V}{T} V_{\text{eff}}(\bar{\phi})\right\} \, , \tag{9.4}$$

where periodic boundary conditions are assumed. We note that the thermodynamic limit $V \to \infty$ is to be taken only after the evaluation of $V_{\text{eff}}(\bar{\phi})$, and that $\int \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \, \phi' = 0$, given that ϕ' by definition only has modes with $\tilde{P} \neq 0$.

In order to carry out the integral in Eq. (9.4), we expand $V_{\text{eff}}(\bar{\phi})$ around its absolute minimum $\bar{\phi}_{\text{min}}$, and perform the corresponding Gaussian integral:

$$V_{\text{eff}}(\bar{\phi}) = V_{\text{eff}}(\bar{\phi}_{\text{min}}) + \frac{1}{2} V_{\text{eff}}''(\bar{\phi}_{\text{min}})(\bar{\phi} - \bar{\phi}_{\text{min}})^2 + \dots \, , \tag{9.5}$$

$$\int_{-\infty}^{\infty} \mathrm{d}\bar{\phi} \, e^{-\frac{V}{T} V_{\text{eff}}(\bar{\phi})} \approx e^{-\frac{V}{T} V_{\text{eff}}(\bar{\phi}_{\text{min}})} \sqrt{\frac{2\pi T}{V_{\text{eff}}''(\bar{\phi}_{\text{min}})V}} \, , \tag{9.6}$$

whereby (we remove reference to μ since there is no chemical potential in this system)

$$f(T) = V_{\text{eff}}(\bar{\phi}_{\text{min}}; T) + \mathcal{O}\left(\frac{\ln V}{V}\right) \, . \tag{9.7}$$

In other words, in the thermodynamic limit $V \to \infty$, the problem reduces to determining V_{eff} and finding its minimum. Note that the value of $\bar{\phi}_{\text{min}}$ depends on the parameters of the problem, in particular on T.

Let us now ask under which conditions a first order phase transition could emerge. We can write

$$\frac{\partial f(T)}{\partial T} = \left[\frac{\partial V_{\text{eff}}(\bar{\phi}; T)}{\partial \bar{\phi}} \frac{\partial \bar{\phi}_{\text{min}}}{\partial T} + \frac{\partial V_{\text{eff}}(\bar{\phi}; T)}{\partial T}\right]_{\bar{\phi}=\bar{\phi}_{\text{min}}} \tag{9.8}$$

$$= \left.\frac{\partial V_{\text{eff}}(\bar{\phi}; T)}{\partial T}\right|_{\bar{\phi}=\bar{\phi}_{\text{min}}} \, , \tag{9.9}$$

where we made use of the fact that $\bar{\phi}_{\text{min}}$ minimizes V_{eff}. Thereby it is clear that $\lim_{T \to T_c^+} \frac{\partial f}{\partial T} \neq \lim_{T \to T_c^-} \frac{\partial f}{\partial T}$ only if $\lim_{T \to T_c^+} \bar{\phi}_{\text{min}} \neq \lim_{T \to T_c^-} \bar{\phi}_{\text{min}}$. In other words, a *first order transition necessitates a discontinuity in $\bar{\phi}_{min}$*:
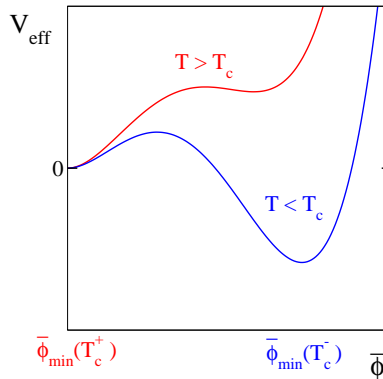
Our task therefore is to evaluate V_{eff}. Before proceeding with the practical computation, let us formulate the generic rules for the computation of the effective potential that can be extracted from the above:

(1) Write $\phi = \bar{\phi} + \phi'$ in $\mathcal{L}_E$.

(2) The part only depending on $\bar{\phi}$ is the 0th order, or tree-level, contribution to V_{eff}.

(3) Any terms linear in ϕ' should be dropped, because $\int \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\, \phi' = 0$.

(4) The remaining contributions to V_{eff} are obtained like $f(T)$ before, except that the masses and couplings of ϕ' now depend on the "shift" $\bar{\phi}$.

(5) However, among all possible graphs, one-particle-reducible graphs (i.e. graphs with a single ϕ'-propagator, the cutting of which would split the graph into two graphs) should be dropped (in addition to disconnected graphs), since such a ϕ'-propagator would necessarily carry zero momentum, which is excluded by definition.

Remarkably, this set of rules is *identical*, from an operational point of view, to the set of rules that can be shown to follow[33] from a totally different (but "standard") definition of an effective potential, namely one based on a Legendre transform of the generating functional:

$$e^{-W[J]} \equiv \int \mathcal{D}\phi\, e^{-S_E - \int_x \phi J}, \tag{9.10}$$

$$\Gamma[\bar{\phi}] \equiv W[J] - J\bar{\phi}, \quad \bar{\phi} \equiv \frac{\delta W[J]}{\delta J}, \tag{9.11}$$

$$V_{\text{eff}}(\bar{\phi}) \equiv \frac{T}{V}\Gamma[\bar{\phi}] \quad \text{for} \quad \bar{\phi} = \text{constant}. \tag{9.12}$$

However, our procedure is actually better than this standard one, because our procedure is defined for any value of $\bar{\phi}$, while the Legendre transform of the standard procedure requires certain properties from the functions in order to be defined, which leads to confusing discussions concerning for instance whether the effective potential necessarily needs to be a convex function.

Let us now proceed with the practical computation. Implementing steps (1) and (2), and indicating terms dropped in step (3) by square brackets, we get

$$\frac{1}{2}\left(\partial_\mu \phi\right)^2 \quad \rightarrow \quad \frac{1}{2}\left(\partial_\mu \phi'\right)^2, \tag{9.13}$$

$$-\frac{1}{2}m^2\phi^2 \quad \rightarrow \quad -\frac{1}{2}m^2\bar{\phi}^2 - \left[m^2\bar{\phi}\phi'\right] - \frac{1}{2}m^2\phi'^2, \tag{9.14}$$

$$\frac{1}{4}\lambda\phi^4 \quad \rightarrow \quad \frac{1}{4}\lambda\bar{\phi}^4 + \left[\lambda\bar{\phi}^3\phi'\right] + \frac{3}{2}\lambda\bar{\phi}^2\phi'^2 + \lambda\bar{\phi}\phi'^3 + \frac{1}{4}\lambda\phi'^4, \tag{9.15}$$

$$\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \quad = \quad \frac{V}{T}, \tag{9.16}$$

$$V_{\text{eff}}^{(0)}(\bar{\phi}) \quad = \quad -\frac{1}{2}m^2\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4. \tag{9.17}$$

Note that, consequently, $V_{\text{eff}}^{(0)}(\bar{\phi})$ is independent of the temperature T.

The dominant "thermal fluctuations" or "radiative corrections" are given by the 1-loop expression. This follows from the part quadratic in ϕ' (since linear terms are dropped). Combining Eqs. (9.14), (9.15), the "effective" mass of ϕ' reads $m_{\text{eff}}^2 \equiv -m^2 + 3\lambda\bar{\phi}^2$, and the corresponding contribution to the effective potential becomes

$$e^{-\frac{V}{T}V_{\text{eff}}^{(1)}} = \int \mathcal{D}\phi'\, e^{-\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x}\, \frac{1}{2}\phi'\left[-\partial_\mu^2 + m_{\text{eff}}^2\right]\phi'} \tag{9.18}$$

$$= \int \mathcal{D}\tilde{\phi}'\, e^{-\frac{T}{V}\sum_{\omega_n,\mathbf{p}}\frac{1}{2}\tilde{\phi}'\left[\omega_n^2 + \mathbf{p}^2 + m_{\text{eff}}^2\right]\tilde{\phi}'} \tag{9.19}$$

$$= \mathcal{C}\left[\prod_{\tilde{P}\neq 0}(\omega_n^2 + \mathbf{p}^2 + m_{\text{eff}}^2)\right]^{-\frac{1}{2}}, \tag{9.20}$$

$$V_{\text{eff}}^{(1)}(\bar{\phi}) = \lim_{V\to\infty}\frac{T}{V}\sum_{\tilde{P}\neq 0}\frac{1}{2}\ln(\omega_n^2 + \mathbf{p}^2 + m_{\text{eff}}^2) - [\text{const.}] \tag{9.21}$$

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[33]R. Jackiw, *Functional Evaluation of the Effective Potential*, Phys. Rev. D 9 (1974) 1686.

In the infinite-volume limit this goes over to the function $J(m_{\mathrm{eff}}, T)$, defined in Eqs. (2.50), (2.51). We return presently to the properties of this function, but let us first specify the way to compute higher-order corrections as well.

Indeed, higher order corrections come from the remaining terms, paying attention to the rules (4) and (5):

$$e^{-\frac{V}{T} V_{\mathrm{eff}}^{(\geq 2)}(\bar{\phi})} = \left\langle \exp\left\{ -S_{E,I}[\bar{\phi}] \right\} \right\rangle_{\mathrm{1PI}} , \tag{9.22}$$

$$S_{E,I}[\bar{\phi}] = \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \left[\lambda \bar{\phi} \phi'^3 + \frac{1}{4} \lambda \phi'^4 \right] , \tag{9.23}$$

where the propagator to be used reads

$$\left\langle \tilde{\phi}'(\tilde{P}) \tilde{\phi}'(\tilde{Q}) \right\rangle = \frac{V}{T} \delta_{\tilde{P}, -\tilde{Q}} \frac{1}{\omega_n^2 + \mathbf{p}^2 + m_{\mathrm{eff}}^2(\bar{\phi})} . \tag{9.24}$$

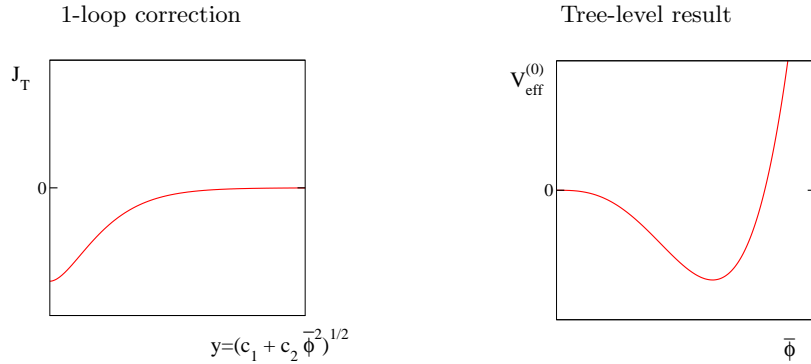We now return to the evaluation of the 1-loop effective potential at $V = \infty$. We obtain

$$V_{\mathrm{eff}}^{(1)} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left[\frac{\omega}{2} + T \ln\left(1 - e^{-\omega/T} \right) \right]_{\omega = \sqrt{\mathbf{p}^2 + m_{\mathrm{eff}}^2}} . \tag{9.25}$$

The temperature-dependent part is as in Eq. (2.58),

$$J_T(m_{\mathrm{eff}}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} T \ln\left[1 - \exp\left(-\frac{\sqrt{\mathbf{p}^2 + m_{\mathrm{eff}}^2}}{T} \right) \right] \tag{9.26}$$

$$= \frac{T^4}{2\pi^2} \int_0^\infty \mathrm{d}x \, x^2 \ln\left[1 - e^{-\sqrt{x^2 + y^2}} \right]_{y = m_{\mathrm{eff}}/T} . \tag{9.27}$$

This function was evaluated in Exercise 3, and we recall that the shape is:

In other words, the symmetric minimum becomes more favoured (has a lower free energy) at high temperatures!

In order to be more quantitative, let us study what happens at very high T. From Eq. (2.81),

$$J_T(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2} \left[\ln\left(\frac{m e^{\gamma_E}}{4\pi T} \right) - \frac{3}{4} \right] + \mathcal{O}(m^6) . \tag{9.28}$$

Keeping just the leading mass-dependent term leads to

$$V_{\mathrm{eff}}^{(0)} + V_{\mathrm{eff}}^{(1)} = [\bar{\phi}\text{-indep.}] + \frac{1}{2} \left(-m^2 + \frac{\lambda}{4} T^2 \right) \bar{\phi}^2 + \frac{1}{4} \lambda \bar{\phi}^4 . \tag{9.29}$$
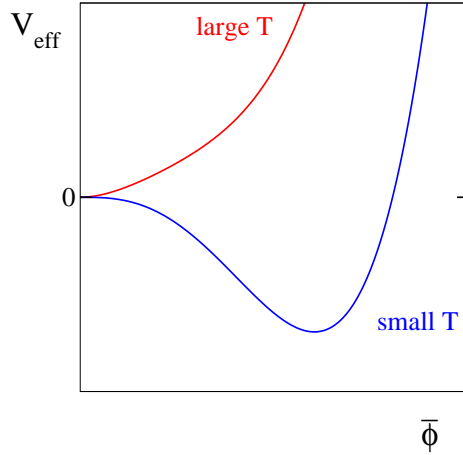
Therefore, for $T \ll 2m/\sqrt{\lambda}$, the symmetry is broken; for $T \gg 2m/\sqrt{\lambda}$, it is restored; and in between there is a phase transition of some kind, as illustrated above.

We may then ask a more refined question, namely, what is the order of the transition? In order to get a first impression on this, let us include the next term from Eq. (9.28). For simplicity, we may even put $m^2 \to 0$, whereby

$$V_{\text{eff}}^{(0)} + V_{\text{eff}}^{(1)} = [\bar{\phi}\text{-indep.}] + \frac{\lambda}{8}T^2\bar{\phi}^2 - \frac{T}{12\pi}(3\lambda)^{3/2}|\bar{\phi}|^3 + \frac{1}{4}\lambda\bar{\phi}^4 \; . \tag{9.30}$$

Thus, it appears like this could be a case of a "fluctuation induced" / "radiatively generated" first order transition:
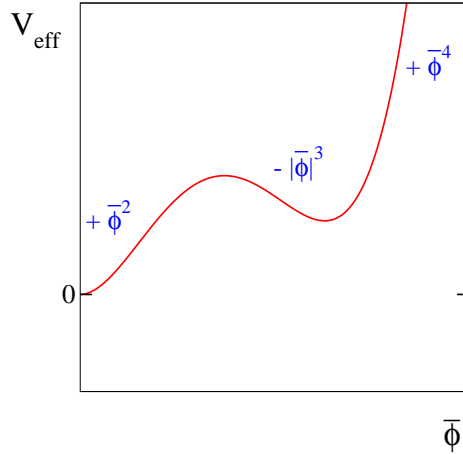
We should not rush to conclusions, however. Indeed, it can be seen from Eq. (9.30) that the broken minimum appears where the cubic and quartic terms are of similar orders of magnitude, i.e.,

$$T\lambda^{\frac{3}{2}}|\bar{\phi}|^3 \sim \lambda|\bar{\phi}|^4 \quad \Rightarrow \quad |\bar{\phi}| \sim \lambda^{\frac{1}{2}}T \; . \tag{9.31}$$

However, the expansion parameter related to higher order corrections, discussed schematically in Sec. 6.1, then becomes

$$\frac{\lambda T}{m_{\text{eff}}} \sim \frac{\lambda T}{\sqrt{3\lambda\bar{\phi}^2}} \sim \frac{\lambda^{\frac{1}{2}}T}{|\bar{\phi}|} \sim \mathcal{O}(1) \; . \tag{9.32}$$

In other words, the perturbative prediction is *not* reliable for the order of the transition.

On the other hand, a reliable analysis can again be carried out with effective field theory techniques, as discussed in Sec. 6.3 for QCD. In the case of the scalar field theory, the dimensionally reduced action takes the form

$$S_{\text{eff}} = \frac{1}{T} \int_V d^3\mathbf{x} \left[\frac{1}{2} \, (\partial_i \phi_3)^2 + \frac{1}{2} \, m_3^2 \phi_3^2 + \frac{1}{4} \, \lambda_3 \phi_3^4 + ... \right] , \tag{9.33}$$

with effective couplings of the form

$$m_3^2 = -m^2[1 + \mathcal{O}(\lambda)] + \frac{1}{4} \lambda T^2 [1 + \mathcal{O}(\lambda)] , \tag{9.34}$$

$$\lambda_3 = \lambda[1 + \mathcal{O}(\lambda)] . \tag{9.35}$$

This system can be studied non-perturbatively (with lattice simulations, ϵ-expansion, etc) to show that there is a *second order transition* at $m_3^2 \approx -m^2 + \lambda T^2/4 \approx 0$, in the 3d Ising universality class.

Finally, we note that if the original theory is more complicated (containing for instance two different fields and coupling constants), then it is possible to arrange the couplings so that the first order signature seen in perturbation theory is physical. For instance:

- A theory with *two real scalar fields* can have a first order transition, if the couplings between the two fields are tuned appropriately[34].

- A theory with *a complex scalar field and U(1) gauge symmetry*, which is just the Ginzburg-Landau theory of superconductivity, does have a first order transition, if the quartic coupling, λ, is small enough compared with the electric coupling squared, e^2.[35]

- The standard electroweak theory, with Higgs doublet(s) + SU(2)×U(1) gauge symmetry, can also have a first order transition if the scalar self-coupling λ is small enough[36].

[34]J. Rudnick, *First-order transition induced by cubic anisotropy*, Phys. Rev. B 18 (1978) 1406.
[35]B.I. Halperin, T.C. Lubensky and S.-K. Ma, *First-Order Phase Transitions in Superconductors and Smectic-A Liquid Crystals*, Phys. Rev. Lett. 32 (1974) 292.
[36]D.A. Kirzhnits and A.D. Linde, *Symmetry Behavior in Gauge Theories*, Annals Phys. 101 (1976) 195.