8.5. Relation to classical kinetic theory

We found in the previous section that the HTL effective theory has the unpleasant feature of being non-local. It turns out, however, that by introducing extra degrees of freedom — hard on-shell particles — it can be recast in a local form, as a "kinetic theory", involving the mentioned particles propagating in a gauge field background. (The "canonical" kinetic theory is defined by the Boltzmann equation; when gauge fields appear as force terms in the Boltzmann equation, and gauge field equations of motion are imposed as a further constraint, the system is often referred to as Vlasov equations.) Such a local reformulation is very useful from the point of view of further analyses, both analytic[25] and numerical[27], and is also conceptually more satisfying than a non-local effective theory. As already mentioned in the previous section, the drawback is that the theory no longer has the appearance of a quantum field theory, whereby some of the familiar tools and results are no longer available.

In order to simplify the analysis a bit, we will start by considering the case of QED, i.e. an Abelian plasma. Then the covariant derivatives in Eq. (8.147) become partial derivatives, and the HTL-correction is quadratic. On the other hand, as we recall from Eq. (7.39), in the Abelian case the effective theory can also possess a term linear in the gauge field A_0, if the fermions carry a non-zero chemical potential.

More precisely, let us assume that the coupling between fermions and gauge fields is mediated by the covariant derivative

$$D_\mu = \partial_\mu - ig A_\mu .$$ (8.148)

Letting

$$j_{\mathrm{E}} \equiv g \frac{\mu}{3} \left(T^2 + \frac{\mu^2}{\pi^2} \right) , \quad m_{\mathrm{E}}^2 \equiv g^2 \left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2} \right) ,$$ (8.149)

and introducing the shorthand notation

$$\int_v \equiv \int \frac{\mathrm{d}\Omega_v}{4\pi}$$ (8.150)

for the velocity integrals, the HTL effective action of Eq. (8.147) can be written as

$$S_M \;=\; \int_{x,v} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_{\mathrm{E}} v^\mu A_\mu - \frac{m_{\mathrm{E}}^2}{4} F_{\alpha\mu} \frac{v^\alpha v^\beta}{(v \cdot \partial)^2} F_\beta{}^\mu \right] .$$ (8.151)

Here we have carried out one "partial integration" in the denominator (this can perhaps easiest be understood by going to momentum space).

Since the action is quadratic in the fields, its contents can equivalently be expressed through equations of motion. It is useful to express the equations of motion in a form following from the identities in Eq. (8.210). We obtain

$$\partial_\mu F^{\mu\nu}(x) \;=\; \int_v \left[j_{\mathrm{E}} v^\nu - m_{\mathrm{E}}^2 \frac{v^\nu v^\alpha}{v \cdot \partial} F_{\alpha 0}(x) \right] .$$ (8.152)

By making use of the properties $\int_v v^\alpha = \delta^\alpha{}_0$ and $F_{00} = 0$ it is easy to see that the right-hand side is divergenceless (or "transverse") with respect to ∂^ν, as it has to be in order for the equation to be consistent.

Let us at this point introduce some notation that will be useful in the following. We denote

$$\delta_+(P^2) \equiv 2\theta(p_0) \, (2\pi) \, \delta(P^2) ,$$ (8.153)

[27]D. Bödeker, G.D. Moore and K. Rummukainen, *Chern-Simons number diffusion and hard thermal loops on the lattice,* Phys. Rev. D 61 (2000) 056003; M. Hindmarsh and A. Rajantie, *Phase transition dynamics in the hot Abelian Higgs model,* Phys. Rev. D 64 (2001) 065016; A. Rebhan, P. Romatschke and M. Strickland, *Dynamics of quark-gluon plasma instabilities in discretized hard-loop approximation,* JHEP 09 (2005) 041.

such that

$$\int_P \delta_+(P^2)\, f(p_0, \mathbf{p}) = \int_{\mathbf{p}} \frac{f(p, \mathbf{p})}{p}, \quad p \equiv |\mathbf{p}| \,. \tag{8.154}$$

All the results will factorise in a form where a phase space integral is left over, which can be carried out explicitly. Let

$$n_{\mathrm{F}}(p) = \frac{1}{e^{\beta p}+1} \,, \quad N_\pm(p) = n_{\mathrm{F}}(p+\mu) \pm n_{\mathrm{F}}(p-\mu) \,. \tag{8.155}$$

Then, irrespective of the relative magnitudes of T, μ,

$$\int_{\mathbf{p}} N_-(p) = -\frac{\mu}{6}\Big(T^2 + \frac{\mu^2}{\pi^2}\Big) \,, \tag{8.156}$$

$$\int_{\mathbf{p}} \frac{N_+(p)}{p} = -\frac{1}{2}\int_{\mathbf{p}} \frac{\partial N_+(p)}{\partial p} = -\frac{1}{2}\frac{\partial}{\partial\mu}\int_{\mathbf{p}} N_-(p) = \frac{1}{4}\Big(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\Big) \,, \tag{8.157}$$

$$\int_{\mathbf{p}} \frac{N_-(p)}{p^2} = -\int_{\mathbf{p}} \frac{1}{p}\frac{\partial N_-(p)}{\partial p} = \frac{1}{2}\int_{\mathbf{p}} \frac{\partial^2 N_-(p)}{\partial p^2} = \frac{1}{2}\frac{\partial^2}{\partial\mu^2}\int_{\mathbf{p}} N_-(p) = -\frac{\mu}{2\pi^2} \,. \tag{8.158}$$

These relations can be proven with the techniques of Exercise 12.

The claim now is that the HTL structures in Eq. (8.152) can be reproduced[28] by classical kinetic theory, or "Vlasov equations". To see this, let us start with classical electrodynamics[29], and define

$$P^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t}, \quad \frac{\mathrm{d}P^\alpha}{\mathrm{d}t} = -g F^\alpha{}_\beta P^\beta. \tag{8.159}$$

Then the collisionless Boltzmann equation for hard particles in a gauge field background becomes

$$\frac{\mathrm{d}f(x,P)}{\mathrm{d}t} = P^\alpha\Big(\frac{\partial f}{\partial x^\alpha} + g F_\alpha{}^\beta \frac{\partial f}{\partial P^\beta}\Big) = 0 \,. \tag{8.160}$$

Let us note that we may in general assume that

$$f(x,P) = \delta_+(P^2)\hat{f}(x,P) \,, \tag{8.161}$$

where $\delta_+(P^2)$ is defined as in Eq. (8.153), since this form is conserved by Eq. (8.160), due to

$$P^\alpha F_\alpha{}^\beta \frac{\partial}{\partial P^\beta}\delta(P^2) = 2P^\alpha P^\beta F_{\alpha\beta}\delta'(P^2) = 0 \,. \tag{8.162}$$

The derivate of $\theta(p_0)$ included in $\delta_+(P^2)$ can be safely ignored as well, since it would contribute only at the point $p_0 = \mathbf{p} = 0$, and has no effect after integration over p.

We formally solve Eq. (8.160) in powers of $gF_{\mu\nu}$: $f = f_0 + f_1 + f_2 + \dots$. This leads to the recursion relation

$$P \cdot \partial f_{n+1}(x,P) = -g P^\alpha F_{\alpha\beta}(x)\frac{\partial f_n(x,P)}{\partial P_\beta} \,. \tag{8.163}$$

The zeroth order gives

$$P \cdot \partial f_0(x,P) = 0 \,. \tag{8.164}$$

We take as a solution a space-time independent function depending, in view of Eq. (8.161), non-trivially only on p_0, parametrized by T, μ, and applying separately to all particle species i:

$$f_0^{(i)} = \delta_+(P^2)\, \hat{f}_0^{(i)}(p_0; T, \mu_i) \equiv \delta_+(P^2)\, n_{\mathrm{F}}(p_0 + \mu_i) \,, \tag{8.165}$$

[28]J.P. Blaizot and E. Iancu, *Kinetic equations for long wavelength excitations of the quark – gluon plasma,* Phys. Rev. Lett. 70 (1993) 3376; P.F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, *Deriving the hard thermal loops of QCD from classical transport theory,* Phys. Rev. Lett. 72 (1994) 3461; F.T. Brandt, J. Frenkel and J.C. Taylor, *High temperature QCD and the classical Boltzmann equation in curved space-time,* Nucl. Phys. B 437 (1995) 433.
[29]For a pedagogic presentation, see R.D. Pisarski, *Nonabelian Debye screening, tsunami waves, and worldline fermions,* lectures at *International School of Astrophysics D. Chalonge,* Erice, Italy, 1997 [hep-ph/9710370].

where $n_F(p_0)$ is from Eq. (8.155). Furthermore, antiparticles are always assumed to come with the opposite signs of g and μ than particles. Thus, a single Dirac fermion contributes two degrees of freedom with $+g, +\mu$, two with $-g, -\mu$.

In addition to these equations, we need the definition of the current induced by the hard particles:

$$j^\mu(x) \equiv -\sum_i g_i \int_P P^\mu f^{(i)}(x, P) \,. \tag{8.166}$$

The equations of motion are

$$S_M^{\text{free}} = -\int_x \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad \frac{\delta}{\delta A_\mu} S_M^{\text{free}} = \partial_\nu F^{\nu\mu} = j^\mu \,. \tag{8.167}$$

The expression for j^μ in terms of the background gauge field thus implies a non-local effective action $S_M = S_M^{\text{free}} + \delta S_M$ for the gauge fields only, where δS_M is to be determined from

$$\frac{\delta}{\delta A_\mu} \delta S_M = -j^\mu \,. \tag{8.168}$$

Let us now work out explicit expressions. We start by considering $f = f_0$. Summing over two degrees of freedom with $+g, +\mu$ and two with $-g, -\mu$, we obtain from Eqs. (8.165), (8.166),

$$j^\mu = -2g \int_P \delta_+(P^2) P^\mu N_-(p_0) = \delta^{\mu 0} g \frac{\mu}{3} \left(T^2 + \frac{\mu^2}{\pi^2} \right) \,, \tag{8.169}$$

where we used Eqs. (8.154), (8.155), (8.156). This indeed agrees with Eq. (8.152).

For the next term we need

$$f_1 = -g \frac{1}{P \cdot \partial} P^\alpha F_{\alpha\beta} \frac{\partial f_0}{\partial P_\beta} \,. \tag{8.170}$$

Inserting into Eq. (8.166),

$$j^\mu(x) = \sum_i g_i^2 \int_P \frac{\partial f_0^{(i)}}{\partial P_\beta} \frac{P^\mu P^\alpha}{P \cdot \partial} F_{\alpha\beta}(x) \,. \tag{8.171}$$

Since f_0 essentially only depends on p^0, we get

$$j^\mu(x) = 2g^2 \underbrace{\int_P \delta_+(P^2) \left[n_F'(p^0 + \mu) + n_F'(p^0 - \mu) \right] p}_{-\frac{1}{2} \left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2} \right)} \int_v \frac{v^\mu v^\alpha}{v \cdot \partial} F_{\alpha 0}(x) \,, \tag{8.172}$$

which indeed agrees with the second term in Eq. (8.152). This completes our proof for the Abelian case.

Let us then move to the non-Abelian case. The formulation will be somewhat more complicated, so we just indicate the form of the kinetic equations, and solve them in the *static limit*.

The simplest way to display the non-Abelian kinetic equations is to follow Eq. (8.160), but replace f by an $N_c \times N_c$ matrix. The collisionless QCD Boltzmann/Vlasov equation for each single fundamentally charged fermionic degree of freedom, and the corresponding gauge current induced, can be written as

$$\left[P \cdot D, f \right] + \frac{g}{2} \left\{ P^\mu F_{\mu\nu}, \frac{\partial f}{\partial P_\nu} \right\} = 0 \,, \tag{8.173}$$

$$j_\mu^a = -g \int_P P_\mu \text{Tr} \left[T^a f \right] \,. \tag{8.174}$$

To express this in a more down-to-earth way, we may write the matrix f in terms of a "singlet" distribution function $\bar{f}$ and an adjoint (or "octet") distribution function f^a:

$$f(x,P) = \frac{1}{N_\mathrm{c}} \bar{f}(x,P) + 2T^a f^a(x,P) \,. \tag{8.175}$$

Making use of

$$\{T^a, T^b\} = \frac{1}{N_\mathrm{c}} \delta^{ab} + d^{abc} T^c \,, \tag{8.176}$$

we can write

$$\left[P \cdot D, f \right] = \frac{1}{N_\mathrm{c}} P \cdot \partial \bar{f} + 2(P \cdot \mathcal{D})^{ab} f^b T^a \,, \tag{8.177}$$

$$\begin{aligned}
\left\{ P^\mu F_{\mu\nu}, \frac{\partial f}{\partial P_\nu} \right\} &= \frac{1}{N_\mathrm{c}} \left\{ P^\mu F_{\mu\nu}, \frac{\partial \bar{f}}{\partial P_\nu} \right\} + 2 P^\mu F_{\mu\nu}^a \frac{\partial f^b}{\partial P_\nu} \{T^a, T^b\} \\
&= \frac{2}{N_\mathrm{c}} P^\mu F_{\mu\nu} \frac{\partial \bar{f}}{\partial P_\nu} + \frac{2}{N_\mathrm{c}} P^\mu F_{\mu\nu}^a \frac{\partial f^a}{\partial P_\nu} + 2 d^{abc} T^c P^\mu F_{\mu\nu}^a \frac{\partial f^b}{\partial P_\nu} \,. \tag{8.178}
\end{aligned}$$

Projecting then Eq. (8.173) with $\mathrm{Tr}\,[...]$ and $\mathrm{Tr}\,[T^a...]$, the equations obtain the forms[30]

$$P \cdot \partial \bar{f} + g P^\mu F_{\mu\nu}^a \frac{\partial f^a}{\partial P_\nu} = 0 \,, \tag{8.179}$$

$$(P \cdot \mathcal{D})^{ab} f^b + \frac{g}{2} d^{abc} P^\mu F_{\mu\nu}^b \frac{\partial f^c}{\partial P_\nu} + \frac{g}{2N_\mathrm{c}} P^\mu F_{\mu\nu}^a \frac{\partial \bar{f}}{\partial P_\nu} = 0 \,, \tag{8.180}$$

$$j_\mu^a = -\sum_i g_i \int_P P_\mu f^{a(i)} \,. \tag{8.181}$$

When computing the current, each quark now comes with N_c colours, in addition to two spin degrees of freedom, both for particles and for anti-particles.

The equations can again be solved iteratively in $gF_{\mu\nu}$: $f = f_0 + f_1 + f_2 + ...$. At the zeroth order,

$$\bar{f}_0^{(i)} = \delta_+(P^2)\, n_\mathrm{F}(p_0 + \mu_i), \tag{8.182}$$

$$f_0^{a(i)} = 0. \tag{8.183}$$

Iterating this, we obtain at the first order,

$$\bar{f}_1^{(i)} = 0 \,, \tag{8.184}$$

$$(P \cdot \mathcal{D})^{ab} f_1^{b(i)} = -\frac{g_i}{2N_\mathrm{c}} P^\mu F_{\mu 0}^a \, \delta_+(P^2)\, n_\mathrm{F}'(p_0 + \mu_i) \,. \tag{8.185}$$

Let us now solve these equations in the static limit. In Eq. (8.185) we can then write

$$P^\mu F_{\mu 0}^a = (P \cdot \mathcal{D} A_0)^a - \partial_0 (P^\mu A_\mu^a) = (P \cdot \mathcal{D})^{ab} A_0^b \,, \tag{8.186}$$

so that

$$f_1^{a(i)} = -\frac{g_i}{2N_\mathrm{c}} A_0^a \, \delta_+(P^2)\, n_\mathrm{F}'(p_0 + \mu_i) \,. \tag{8.187}$$

Inserting into Eq. (8.181), we obtain

$$\begin{aligned}
j_\mu^a = -\sum_i g_i \int_P P_\mu f_1^{a(i)} &= g^2 A_0^a \int_P P_\mu \, \delta_+(P^2)\, N_+'(p_0) \\
&= \delta_{\mu 0}\, g^2 A_0^a \int_\mathbf{P} N_+'(p) \,. \tag{8.188}
\end{aligned}$$

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[30]U. Heinz, *Kinetic Theory For Nonabelian Plasmas,* Phys. Rev. Lett. 51 (1983) 351; H. Elze and U. Heinz, *Quark - Gluon Transport Theory,* Phys. Rept. 183 (1989) 81.

After use of Eq. (8.157), and a proper account of the signs between $\mathcal{L}_E$ and $\mathcal{L}_M$ as well as between Euclidean and Minkowskian A_0, this indeed agrees with the derivative of the mass term in Eq. (6.36).

Let us count the amount of information that needed to be added to the HTL action, in order to make it local. In full generality, the classical distribution functions $f(x, P)$ depend on (4+4) coordinates. As we have discussed, $f(x, P) = \delta_+(P^2)\hat{f}(x, P)$, i.e. the hard particles can be set on-shell; thus the dependence can thus be reduced to, say, the spatial components $\mathbf{p}$. However, one further simplification is possible in Eq. (8.185), by writing

$$f_1^{a(i)} = -\frac{g_i}{2N_{\mathrm{c}}}\,\delta_+(P^2)\,n_{\mathrm{F}}'(p_0 + \mu_i)\,W^a(x, \mathbf{v}) \;, \tag{8.189}$$

so that

$$(v \cdot \mathcal{D})^{ab}W^b(x, \mathbf{v}) = v^\mu F_{\mu 0}^a(x) \;. \tag{8.190}$$

We can also perform the sum over i and the integral over p_0 in Eq. (8.181), reducing thus the dependence only to angular variables and a total of (4+2) dimensions. Nevertheless, we needed to introduce *extra dimensions*, not only extra degrees of freedom, in order to make the HTL theory local!

As a final remark we note that the equations written down so far did not specify what kind of initial conditions should be assumed for the gauge fields. Indeed, in order to have a proper statistical weighting over the initial conditions for the time evolution, one should also work out a Hamiltonian formulation in terms of the gauge fields A_i^a, $E_i^a \equiv F_{0i}^a$ and W^a; afterwards one can weigh by $\exp(-H/T)$. This issue is not altogether trivial but does turn out to possess a solution[31].

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[31] J.P. Blaizot and E. Iancu, *Soft collective excitations in hot gauge theories,* Nucl. Phys. B 417 (1994) 608; V.P. Nair, *Hard thermal loops, gauged WZNW action and the energy of hot quark - gluon plasma,* Phys. Rev. D 48 (1993) 3432 *Hamiltonian analysis of the effective action for hard thermal loops in QCD,* Phys. Rev. D 50 (1994) 4201.

8.6. Exercise 12

Compute the radial and angular integrals in Eqs. (8.115)–(8.118).

Solution to Exercise 12

Eq. (8.115) can be verified by straightforward partial integration:

$$\int_0^\infty \mathrm{d}s \, \frac{\mathrm{d}s}{\mathrm{d}s} \, s \left[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \right] = -\int_0^\infty \mathrm{d}s \, s \left[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \right]$$
$$- \int_0^\infty \mathrm{d}s \, s^2 \left[n_\mathrm{F}'(s-\mu) + n_\mathrm{F}'(s+\mu) \right] . \quad (8.191)$$

Moving the first term on the right-hand side to the left-hand side leads directly to Eq. (8.115).

Eq. (8.116) can be verified for instance by starting from a combination of Eqs. (7.35), (7.41):

$$-f(T,\mu) = 2\int_s \left\{ s + T \left[\ln\left(1 + e^{-\frac{s+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{s-\mu}{T}}\right) \right] \right\}$$
$$= \frac{7\pi^2 T^4}{180} + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2} . \quad (8.192)$$

Taking the second partial derivative with respect to μ, we get

$$-\frac{\partial^2 f(T,\mu)}{\partial \mu^2} = 2\int_s \left\{ T \frac{\partial^2}{\partial \mu^2} \left[\ln\left(1 + e^{-\frac{s+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{s-\mu}{T}}\right) \right] \right\}$$
$$= 2\int_s \left\{ T \frac{\mathrm{d}^2}{\mathrm{d}s^2} \left[\ln\left(1 + e^{-\frac{s+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{s-\mu}{T}}\right) \right] \right\}$$
$$= 4T \int_s \frac{1}{s^2} \left[\ln\left(1 + e^{-\frac{s+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{s-\mu}{T}}\right) \right] \quad (8.193)$$
$$= \frac{T^2}{3} + \frac{\mu^2}{\pi^2} , \quad (8.194)$$

where in the penultimate step we carried out two partial integrations.

On the other hand, the integral in Eq. (8.193) can be rewritten as

$$\int_s \frac{1}{s^2} \left[\ln\left(1 + e^{-\frac{s+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{s-\mu}{T}}\right) \right]$$
$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}s \, \frac{\mathrm{d}s}{\mathrm{d}s} \left[\ln\left(1 + e^{-\frac{s+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{s-\mu}{T}}\right) \right]$$
$$= -\frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}s \, s \left[\frac{e^{-\frac{s+\mu}{T}}}{1 + e^{-\frac{s+\mu}{T}}} + \frac{e^{-\frac{s-\mu}{T}}}{1 + e^{-\frac{s-\mu}{T}}} \right] \left(-\frac{1}{T}\right)$$
$$= \frac{1}{T} \int_s \frac{1}{s} \left[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \right] . \quad (8.195)$$

Replacing the integral in Eq. (8.193) by that in Eq. (8.195) leads then to Eq. (8.116).

Eq. (8.117) is a trivial consequence of rotational symmetry, and the fact that $\mathbf{v}^2 = 1$.

As far as Eq. (8.118) goes, we start by carrying out a simpler integral:

$$L \equiv \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{1}{i\tilde{q}_0 - \mathbf{q} \cdot \mathbf{v}} = \frac{1}{4\pi} 2\pi \int_{-1}^{+1} \mathrm{d}z \frac{1}{i\tilde{q}_0 - |\mathbf{q}|z}$$

$$= -\frac{1}{2|\mathbf{q}|} \int_{-1}^{+1} \mathrm{d}z \, \frac{\mathrm{d}}{\mathrm{d}z} \ln(i\tilde{q}_0 - |\mathbf{q}|z)$$

$$= \frac{1}{2|\mathbf{q}|} \ln \frac{i\tilde{q}_0 + |\mathbf{q}|}{i\tilde{q}_0 - |\mathbf{q}|} \; . \tag{8.196}$$

Further integrals can then be obtained by making use of rotational symmetry: for instance,

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v_i}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} = q_i \, f(i\tilde{q}_0, |\mathbf{q}|) \; , \tag{8.197}$$

where, contracting both sides with $\mathbf{q}$,

$$f(i\tilde{q}_0, |\mathbf{q}|) = \frac{1}{\mathbf{q}^2} \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{\mathbf{q}\cdot\mathbf{v}}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} = \frac{1}{\mathbf{q}^2} \left[-1 + i\tilde{q}_0 \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{1}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \right] . \tag{8.198}$$

Another trick, needed for having higher powers in the denominator, is to take derivatives of Eq. (8.196) with respect to $i\tilde{q}_0$.

Without carrying out any further steps, we list finally the results for a number of velocity integrals that can be obtained this way. Let us change the notation a bit at this point: we now replace $i\tilde{q}_0$ by $q^0 + i0^+$, as is relevant for retarded Green's functions ($i0^+$ is not shown explicitly), and introduce the light-like four-velocity $v \equiv (1, \mathbf{v})$. Then the integrals read ($i, j = 1, 2, 3$)

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} = 1 \; , \tag{8.199}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} v^i = 0 \; , \tag{8.200}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} v^i v^j = \frac{1}{3} \delta^{ij} \; , \tag{8.201}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{1}{v\cdot Q} = L(Q) \; , \tag{8.202}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^i}{v\cdot Q} = \frac{q^i}{|\mathbf{q}|^2} \left[-1 + q^0 L(Q) \right] , \tag{8.203}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^i v^j}{v\cdot Q} = \frac{L(Q)}{2} \left(\delta^{ij} - \frac{q^i q^j}{|\mathbf{q}|^2} \right) + \frac{q^0}{2|\mathbf{q}|^2} \left[1 - q^0 L(Q) \right] \left(\delta^{ij} - 3\frac{q^i q^j}{|\mathbf{q}|^2} \right) , \tag{8.204}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{1}{(v\cdot Q)^2} = \frac{1}{Q^2} \; , \tag{8.205}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^i}{(v\cdot Q)^2} = \frac{q^i}{|\mathbf{q}|^2} \left[\frac{q^0}{Q^2} - L(Q) \right] , \tag{8.206}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^i v^j}{(v\cdot Q)^2} = \frac{1}{2Q^2} \left(\delta^{ij} - \frac{q^i q^j}{|\mathbf{q}|^2} \right) - \frac{1}{2|\mathbf{q}|^2} \left[1 - 2q^0 L(Q) + \frac{(q^0)^2}{Q^2} \right] \left(\delta^{ij} - 3\frac{q^i q^j}{|\mathbf{q}|^2} \right) , \tag{8.207}$$

where $v^\mu \equiv (1, v^i)$, $Q \equiv (q^0, \mathbf{q})$, our metric convention is $(+---)$, $v\cdot Q = q^0 - \mathbf{v}\cdot\mathbf{q}$, and

$$L(Q) \equiv \frac{1}{2|\mathbf{q}|} \ln \frac{q^0 + |\mathbf{q}|}{q^0 - |\mathbf{q}|} \approx -\frac{i\pi}{2|\mathbf{q}|} + \frac{q^0}{|\mathbf{q}|^2} + \frac{(q^0)^3}{3|\mathbf{q}|^4} + \dots \; . \tag{8.208}$$

The following identities (which can be derived by tedious explicit use of the relations obtained) are sometimes very useful:

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^\alpha Q^\beta}{(v\cdot Q)^2} \epsilon_{\alpha\beta\mu\nu} I^\mu J^\nu = \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^\alpha}{v\cdot Q} \epsilon_{0\alpha\mu\nu} I^\mu J^\nu \; , \tag{8.209}$$

$$\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^\alpha v^\beta}{(v\cdot Q)^2} Q_{[\alpha} I_{\mu]} Q_{[\beta} J_{\nu]} \eta^{\mu\nu} = 2 \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^\alpha v^\beta}{v\cdot Q} I_\alpha Q_{[\beta} J_{0]} = 2 \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v^\alpha v^\beta}{v\cdot Q} Q_{[\alpha} I_{0]} J_\beta \; . \tag{8.210}$$

Here $Q_{[\alpha} I_{\mu]} \equiv Q_\alpha I_\mu - Q_\mu I_\alpha$, and I, J are arbitrary Lorentz vectors.