8.4. Hard Thermal Loop effective theory

In the case of static observables, we realized in Sec. 3.5 that the perturbative series suffers in general from serious infrared divergences. However, as discussed in Sec. 6.1, these divergences can only be associated with bosonic Matsubara zero modes. They can therefore be cleanly isolated by constructing an effective field theory for the bosonic Matsubara zero modes, as we did in Sec. 6.3.

The situation is somewhat more complicated in the case of real-time observables. Indeed, as Eq. (8.27) shows, the dependence on *all* the Matsubara modes is needed in order to carry out the analytic continuation leading to the spectral function, even if we were only interested in its behaviour at small frequencies $|q^0| \ll T$. (The same holds also in the opposite direction: as the sum rule in Eq. (8.26) shows, the information contained in the Matsubara zero mode is spread out to *all* q^0's in the Minkowskian formulation.) Therefore, it is not necessarily easy to isolate the soft/light degrees of freedom for which to write down the most general effective Lagrangian.

Nevertheless, it turns out that the Dimensionally Reduced effective field theory of Sec. 6.3 *can* to some extent be generalized to apply to real-time observables as well. In the case of QCD, the generalization is known as the *Hard Thermal Loop* effective theory. The effective theory dictates what kind of resummed propagators should be used for instance in the computation of Sec. 8.2 in order to generate a systematic weak-coupling series. Since the issue is technically relatively complicated we will, however, restrict ourselves to gluons here, the goal being then essentially to generalize the analysis of Sec. 5.5 to a real-time situation.

More precisely, Hard Thermal Loops (HTL) can operationally be defined as follows[22]:

- Consider "soft" external frequencies and momenta, $|q^0|, |\mathbf{q}| \lesssim gT$.
- Inside the loops, sum over all Matsubara frequencies ω_n.
- Subsequently, integrate over "hard" spatial loop momenta, $|\mathbf{s}| \gtrsim T$, Taylor-expanding to leading non-trivial order in $|q^0|/|\mathbf{s}|$, $|\mathbf{q}|/|\mathbf{s}|$.

In order to illustrate the procedure, let us compute the gluon self-energy in this situation. The computation is much like that in Sec. 5.5, except that now we need to keep the external momentum ($\tilde{Q}$) non-zero while carrying out the Matsubara sum, because the full dependence on $\tilde{q}_0$ is needed in order to be able to carry out the analytic continuation; we can take $q^0, \mathbf{q}$ to be soft only *after the analytic continuation.*

For simplicity, we will focus on the quark loop for now. The contributions of the gluon and ghost loops can be analysed in precisely the same way, and in the end they manifest themselves through an additional prefactor precisely like in the parameter $m_\mathrm{E}^2 = g^2T^2(\frac{N_\mathrm{c}}{3} + \frac{N_\mathrm{f}}{6})$ in Eq. (5.96).

As a starting point, we then take Eq. (5.90), interpreted as a self-energy contribution, $\Pi_{\mu\nu}(\tilde{Q})$. Furthermore, we restrict to the spatial part, Π_{ij}, for the moment. Taking also the high-temperature limit $T \gg m$, Eq. (5.90) can be rewritten as

$$
\begin{aligned}
\Pi_{ij}(\tilde{Q}) &= -2g^2 N_\mathrm{f} \SumInt_{\{\tilde{S}_\mathrm{f}\}} \frac{\delta_{ij}(\tilde{S}^2 - \tilde{Q}\cdot\tilde{S}) - 2\tilde{S}_i\tilde{S}_j + \tilde{Q}_i\tilde{S}_j + \tilde{Q}_j\tilde{S}_i}{\tilde{S}^2(\tilde{Q}-\tilde{S})^2} \\
&= -2g^2 N_\mathrm{f} \SumInt_{\{\tilde{S}_\mathrm{f}\}} \frac{\delta_{ij}\left(\frac{1}{2}(\tilde{Q}-\tilde{S})^2 + \frac{1}{2}\tilde{S}^2 - \frac{1}{2}\tilde{Q}^2\right) - 2\tilde{S}_i\tilde{S}_j + \tilde{Q}_i\tilde{S}_j + \tilde{Q}_j\tilde{S}_i}{\tilde{S}^2(\tilde{Q}-\tilde{S})^2}
\end{aligned}
$$

[22] R.D. Pisarski, *Scattering Amplitudes in Hot Gauge Theories,* Phys. Rev. Lett. 63 (1989) 1129; J. Frenkel and J.C. Taylor, *High Temperature Limit of Thermal QCD,* Nucl. Phys. B 334 (1990) 199; E. Braaten and R.D. Pisarski, *Soft Amplitudes in Hot Gauge Theories: a General Analysis,* Nucl. Phys. B 337 (1990) 569; J.C. Taylor and S.M.H. Wong, *The Effective Action of Hard Thermal Loops in QCD,* Nucl. Phys. B 346 (1990) 115.

$$= \quad -2g^2 N_{\rm f} \int_{\mathbf{s}} T \sum_{\tilde{s}_{\rm of}} \bigg[\frac{\delta_{ij}}{\tilde{S}^2} + \frac{1}{\tilde{S}^2(\tilde{Q}-\tilde{S})^2} \bigg(-\frac{\delta_{ij}}{2}\tilde{Q}^2 - 2 s_i s_j + q_i s_j + q_j s_i \bigg) \bigg] \,. \quad (8.101)$$

Here, for generality, we can assume that $\tilde{s}_{0\rm f} = \omega_{\rm f} - i\mu$.

The Matsubara sums can now be carried out. Denoting

$$E_1 \equiv |\mathbf{s}| \,, \quad E_2 \equiv |\mathbf{s}-\mathbf{q}| \,, \quad (8.102)$$

we can read from Eq. (8.63) that

$$\begin{aligned} T\sum_{\omega_{\rm f}} \frac{1}{(\omega_{\rm f}-i\mu)^2+E_1^2} &= \frac{1}{2E_1}\Big[n_{\rm F}(E_1+\mu)e^{\beta(E_1+\mu)} - n_{\rm F}(E_1-\mu) \Big] \\ &= \frac{1}{2E_1}\Big[1 - n_{\rm F}(E_1+\mu) - n_{\rm F}(E_1-\mu) \Big] \,. \end{aligned} \quad (8.103)$$

It is somewhat more tedious to carry out the other sum. We proceed in analogy with the analysis of Eq. (8.74). Denoting the result by $\mathcal{G}$, we get

$$\begin{aligned} \mathcal{G} &= T\sum_{\tilde{s}_{\rm of}} \frac{1}{[\tilde{s}_{0\rm f}^2+E_1^2][(\tilde{q}_{0\rm b}-\tilde{s}_{0\rm f})^2+E_2^2]} \\ &= T\sum_{\tilde{s}_{\rm of}} T\sum_{\tilde{r}_{\rm of}} \beta\delta(\tilde{r}_{0\rm f}+\tilde{q}_{0\rm b}-\tilde{s}_{0\rm f}) \frac{1}{[\tilde{s}_{0\rm f}^2+E_1^2][\tilde{r}_{0\rm f}^2+E_2^2]} \\ &= \int_0^\beta {\rm d}\tau\, e^{i\tilde{q}_{0\rm b}\tau} \bigg\{ T\sum_{\tilde{s}_{\rm of}} \frac{e^{-i\tilde{s}_{\rm of}\tau}}{\tilde{s}_{0\rm f}^2+E_1^2} \bigg\} \bigg\{ T\sum_{\tilde{r}_{\rm of}} \frac{e^{i\tilde{r}_{\rm of}\tau}}{\tilde{r}_{0\rm f}+E_2^2} \bigg\} \,, \end{aligned} \quad (8.104)$$

where we proceeded like in Eq. (8.76). The sums can be carried out by making use of Eq. (8.63):

$$T\sum_{\tilde{r}_{\rm of}} \frac{e^{i\tilde{r}_{\rm of}\tau}}{\tilde{r}_{0\rm f}^2+E_2^2} = \frac{1}{2E_2}\Big[n_{\rm F}(E_2+\mu)e^{(\beta-\tau)E_2+\beta\mu} - n_{\rm F}(E_2-\mu)e^{\tau E_2} \Big] \,, \quad (8.105)$$

$$\begin{aligned} T\sum_{\tilde{s}_{\rm of}} \frac{e^{-i\tilde{s}_{\rm of}\tau}}{\tilde{s}_{0\rm f}^2+E_1^2} &= -e^{-\mu\beta} T\sum_{\tilde{s}_{\rm of}} \frac{e^{i\tilde{s}_{\rm of}(\beta-\tau)}}{\tilde{s}_{0\rm f}^2+E_1^2} \\ &= \frac{1}{2E_1}\Big[n_{\rm F}(E_1-\mu)e^{(\beta-\tau)E_1-\beta\mu} - n_{\rm F}(E_1+\mu)e^{\tau E_1} \Big] \,, \end{aligned} \quad (8.106)$$

where in the latter equation attention needed to be paid to the fact that Eq. (8.63) only applies for $0 \le \tau \le \beta$ and that there is a shift due to the chemical potential in $\tilde{s}_{0\rm f}$.

Inserting into Eq. (8.104) and carrying out the integral over τ, we get

$$\begin{aligned} \mathcal{G} &= \int_0^\beta {\rm d}\tau\, e^{i\tilde{q}_{0\rm b}\tau} \frac{1}{4E_1E_2} \bigg\{ n_{\rm F}(E_1-\mu)n_{\rm F}(E_2+\mu)e^{(\beta-\tau)(E_1+E_2)} \\ &\qquad - n_{\rm F}(E_1-\mu)n_{\rm F}(E_2-\mu)e^{\tau(E_2-E_1)+\beta(E_1-\mu)} \\ &\qquad - n_{\rm F}(E_1+\mu)n_{\rm F}(E_2+\mu)e^{\tau(E_1-E_2)+\beta(E_2+\mu)} \\ &\qquad + n_{\rm F}(E_1+\mu)n_{\rm F}(E_2-\mu)e^{\tau(E_1+E_2)} \bigg\} \\ &= \frac{1}{4E_1E_2} \bigg\{ n_{\rm F}(E_1-\mu)n_{\rm F}(E_2+\mu) \frac{1}{i\tilde{q}_{0\rm b}-E_1-E_2} \Big[1 - e^{\beta(E_1+E_2)} \Big] \\ &\qquad - n_{\rm F}(E_1-\mu)n_{\rm F}(E_2-\mu) \frac{1}{i\tilde{q}_{0\rm b}+E_2-E_1} \Big[e^{\beta(E_2-\mu)} - e^{\beta(E_1-\mu)} \Big] \end{aligned}$$

$$
\begin{aligned}
& -n_{\rm F}(E_1+\mu)n_{\rm F}(E_2+\mu)\frac{1}{i\tilde{q}_{0\rm b}+E_1-E_2}\Big[e^{\beta(E_1+\mu)}-e^{\beta(E_2+\mu)}\Big] \\
& +n_{\rm F}(E_1+\mu)n_{\rm F}(E_2-\mu)\frac{1}{i\tilde{q}_{0\rm b}+E_1+E_2}\Big[e^{\beta(E_1+E_2)}-1\Big]\bigg\} \\
= \; & \frac{1}{4E_1E_2}\bigg\{\frac{1}{i\tilde{q}_{0\rm b}-E_1-E_2}\Big[n_{\rm F}(E_1-\mu)+n_{\rm F}(E_2+\mu)-1\Big] \\
& +\frac{1}{i\tilde{q}_{0\rm b}+E_2-E_1}\Big[n_{\rm F}(E_2-\mu)-n_{\rm F}(E_1-\mu)\Big] \\
& +\frac{1}{i\tilde{q}_{0\rm b}+E_1-E_2}\Big[n_{\rm F}(E_1+\mu)-n_{\rm F}(E_2+\mu)\Big] \\
& +\frac{1}{i\tilde{q}_{0\rm b}+E_1+E_2}\Big[1-n_{\rm F}(E_1+\mu)-n_{\rm F}(E_2-\mu)\Big]\bigg\}\;.
\end{aligned} \tag{8.107}
$$

At this point we could carry out the analytic continuation $i\tilde{q}_{0\rm b}\to q^0+i0^+$, but it will be convenient to postpone it for a moment; we just have to keep in mind that after the analytic continution, $i\tilde{q}_{0\rm b}$ becomes a *soft* quantity.

The next step is a Taylor-expansion to leading order in $q^0,\mathbf{q}$. In fact, we will be satisfied with the leading term, of $\mathcal{O}(1)$. We can then write

$$
E_1 = s\equiv|\mathbf{s}|\;,\quad E_2=|\mathbf{s}-\mathbf{q}|\approx s-q_i\frac{\partial}{\partial s_i}|\mathbf{s}| = s-q_iv_i\;, \tag{8.108}
$$

where

$$
v_i\equiv\frac{s_i}{s}\;,\quad i=1,2,3\;, \tag{8.109}
$$

are referred to as the *velocities of the hard particles*.

The non-trivial terms in the Taylor-expansion are the 2nd and the 3rd ones in Eq. (8.107). Furthermore, it has to be realized that a Taylor-expansion is sensible only in terms where there is a *scale* guaranteeing that s is hard, $s\sim T$. This can only happen in the presence of the thermal distribution functions. Hence we cannot carry out a Taylor-expansion in the vacuum part; it could be separately verified that this part vanishes for small $q^0,\mathbf{q}$ (there is no gluon mass at zero temperature!), and here we simply omit the zero-temperature part. With these approximations,

$$
\begin{aligned}
\mathcal{G} \;\approx\; & \frac{1}{4s^2}\bigg\{\frac{1}{2s}\Big[-n_{\rm F}(s-\mu)-n_{\rm F}(s+\mu)\Big] \\
& +\frac{1}{i\tilde{q}_{0\rm b}-\mathbf{q}\cdot\mathbf{v}}(-\mathbf{q}\cdot\mathbf{v})n'_{\rm F}(s-\mu) \\
& +\frac{1}{i\tilde{q}_{0\rm b}+\mathbf{q}\cdot\mathbf{v}}(+\mathbf{q}\cdot\mathbf{v})n'_{\rm F}(s+\mu) \\
& +\frac{1}{2s}\Big[-n_{\rm F}(s+\mu)-n_{\rm F}(s-\mu)\Big]\bigg\}+\mathcal{O}(q^0,\mathbf{q})\;.
\end{aligned} \tag{8.110}
$$

Now we can insert Eqs. (8.103), (8.110) into Eq. (8.101). Through the substitution $\mathbf{s}\to-\mathbf{s}$ (whereby $\mathbf{v}\to-\mathbf{v}$), the 3rd row in Eq. (8.110) can be put in the same form as the 2nd row. Furthermore, terms containing $\tilde{q}_0$ or $\mathbf{q}$ in the numerator in Eq. (8.101) are seen to be of higher order. Thereby

$$
\begin{aligned}
\Pi_{ij}(\tilde{Q}) \;\approx\; & -2g^2N_{\rm f}\int_{\mathbf{s}}\bigg\{\frac{\delta_{ij}}{2s}\Big[-n_{\rm F}(s-\mu)-n_{\rm F}(s+\mu)\Big] \\
& -\frac{2s_is_j}{4s^2}\frac{1}{s}\Big[-n_{\rm F}(s-\mu)-n_{\rm F}(s+\mu)\Big] \\
& -\frac{2s_is_j}{4s^2}\frac{i\tilde{q}_{0\rm b}-\mathbf{q}\cdot\mathbf{v}-i\tilde{q}_{0\rm b}}{i\tilde{q}_{0\rm b}-\mathbf{q}\cdot\mathbf{v}}\Big[n'_{\rm F}(s-\mu)+n'_{\rm F}(s+\mu)\Big]\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -2g^2 N_\mathrm{f} \int_\mathbf{s} \bigg\{ \frac{-\delta_{ij}}{2s} \Big[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \Big] \\
& + \frac{v_i v_j}{2s} \Big[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \Big] \\
& - \frac{v_i v_j}{2} \Big[n_\mathrm{F}'(s-\mu) + n_\mathrm{F}'(s+\mu) \Big] \\
& + \frac{v_i v_j i\tilde{q}_0}{2(i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v})} \Big[n_\mathrm{F}'(s-\mu) + n_\mathrm{F}'(s+\mu) \Big] \bigg\} \; . \qquad (8.111)
\end{aligned}
$$

The integrals appearing here are finite, and can be carried out with $d = 3$. We can write the measure as

$$
\int_\mathbf{s} = \int \frac{\mathrm{d}^3\mathbf{s}}{(2\pi)^3} = \int_s \int \frac{\mathrm{d}\Omega_v}{4\pi} \; , \qquad (8.112)
$$

where the *radial integration* reads

$$
\int_s \equiv \frac{4\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}s \, s^2 \; , \qquad (8.113)
$$

while the *angular integration* goes over the directions of $\mathbf{v}$, and is normalized to unity:

$$
\int \frac{\mathrm{d}\Omega_v}{4\pi} = 1 \; . \qquad (8.114)
$$

Then, the following identities can be verified (Exercise 12):

$$
\begin{aligned}
\int_s \Big[n_\mathrm{F}'(s-\mu) + n_\mathrm{F}'(s+\mu) \Big] &= -2 \int_s \frac{1}{s} \Big[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \Big] \; , \qquad (8.115) \\
\int_s \frac{1}{s} \Big[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \Big] &= \frac{1}{4} \bigg(\frac{T^2}{3} + \frac{\mu^2}{\pi^2} \bigg) \; , \qquad (8.116) \\
\int \frac{\mathrm{d}\Omega_v}{4\pi} v_i v_j &= \frac{\delta_{ij}}{3} \; . \qquad (8.117)
\end{aligned}
$$

The integration

$$
\int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v_i v_j}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \qquad (8.118)
$$

can also be carried out (Exercise 12) but we do not need its value for the moment.

With these ingredients, Eq. (8.111) becomes

$$
\begin{aligned}
\Pi_{ij}(\tilde{Q}) \quad \approx \quad & -2g^2 N_\mathrm{f} \int_s \frac{1}{s} \Big[n_\mathrm{F}(s-\mu) + n_\mathrm{F}(s+\mu) \Big] \\
\times \quad & \bigg\{ \delta_{ij} \bigg(-\frac{1}{2} + \frac{1}{6} + \frac{1}{3} \bigg) - \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v_i v_j i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \bigg\} \\
= \quad & g^2 N_\mathrm{f} \bigg(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2} \bigg) \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v_i v_j i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \; . \qquad (8.119)
\end{aligned}
$$

Including also gluons and ghosts, the complete result reads

$$
\Pi_{ij}(\tilde{Q}) = m_\mathrm{E}^2 \int \frac{\mathrm{d}\Omega_v}{4\pi} \frac{v_i v_j i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} + \mathcal{O}(\tilde{q}_0, \mathbf{q}) \; , \qquad (8.120)
$$

where m_E is the generalization of Eq. (5.96) to the case of a fermionic chemical potential,

$$
m_\mathrm{E}^2 \equiv g^2 \bigg[N_\mathrm{c} \frac{T^2}{3} + N_\mathrm{f} \bigg(\frac{T^2}{6} + \frac{\mu^2}{2\pi^2} \bigg) \bigg] \; . \qquad (8.121)
$$

Eq. (8.120), known for the case of QED since a long time[23], is quite a remarkable expression. Even though it is of $\mathcal{O}(1)$ is we count $i\tilde{q}_0$ and $\mathbf{q}$ as quantities of the same order, it depends very non-trivially on the ratio $i\tilde{q}_0/|\mathbf{q}|$. In particular, for $q^0 = i\tilde{q}_0 \to 0$, i.e. *in the static limit*, Π_{ij} obviously vanishes. This corresponds to the result that we obtained in Eq. (5.94). On the other hand, for $0 < |q^0| < |\mathbf{q}|$, it contains both a real and an imaginary part (cf. Eqs. (8.200), (8.204)). The imaginary part is related to the physics of *Landau damping*: it means that nearly static gauge fields can lose energy to the hard particle degrees of freedom in the plasma.

So far, we were only concerned with the spatial self-energy correction Π_{ij}. An interesting question now is to generalize the computation to the full $\Pi_{\mu\nu}$. Fortunately, it turns out that all the information needed can be extracted from Eq. (8.120), as we now show.

Indeed, the self-energy $\Pi_{\mu\nu}$, obtained by integrating out the hard modes, must produce a structure which is gauge-invariant in "soft" gauge transformations; therefore it must be *transverse*. However, what we may mean by transversality changes from the case of zero temperature, because the heat bath introduces a preferred frame, and thus breaks Lorentz invariance. More precisely, we can now introduce *two different* projection operators,

$$P^T_{00}(\tilde{Q}) = P^T_{0i}(\tilde{Q}) = P^T_{i0}(\tilde{Q}) \equiv 0 \,, \quad P^T_{ij}(\tilde{Q}) \equiv \delta_{ij} - \frac{\tilde{q}_i\tilde{q}_j}{\tilde{\mathbf{q}}^2} \,, \tag{8.122}$$

$$P^E_{\mu\nu}(\tilde{Q}) \equiv \delta_{\mu\nu} - \frac{\tilde{q}_\mu\tilde{q}_\nu}{\tilde{Q}^2} - P^T_{\mu\nu}(\tilde{Q}) \,, \tag{8.123}$$

which both are transverse:

$$P^T_{\mu\nu}\tilde{Q}_\nu = P^E_{\mu\nu}\tilde{Q}_\nu = 0 \,. \tag{8.124}$$

Also, the two projectors are transverse to each other, $P^E_{\mu\alpha}P^T_{\alpha\nu} = 0$; this can be seen for instance by choosing a frame where $\mathbf{q} = q\,\mathbf{e}_3$; then

$$P^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,, \quad P^E = \frac{1}{\tilde{Q}^2}\begin{pmatrix} q^2 & 0 & 0 & -q\tilde{q}_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -q\tilde{q}_0 & 0 & 0 & \tilde{q}_0^2 \end{pmatrix} \,. \tag{8.125}$$

With these projectors, we can write

$$\Pi_{ij}(\tilde{Q}) = m_{\rm E}^2 \int \frac{{\rm d}\Omega_v}{4\pi} \frac{v_i v_j i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \equiv P^T_{ij}(\tilde{Q})\,\Pi_T(\tilde{Q}) + P^E_{ij}(\tilde{Q})\,\Pi_E(\tilde{Q}) \,. \tag{8.126}$$

Contracting Eq. (8.126) with δ_{ij} and with q_iq_j leads to the equations

$$m_{\rm E}^2 i\tilde{q}_0 L = 2\Pi_T + \left(1 - \frac{q^2}{\tilde{q}_0^2 + q^2}\right)\Pi_E = 2\Pi_T + \frac{\tilde{q}_0^2}{\tilde{q}_0^2 + q^2}\Pi_E \,, \tag{8.127}$$

$$m_{\rm E}^2 \int \frac{{\rm d}\Omega_v}{4\pi} \frac{(\mathbf{q}\cdot\mathbf{v})^2 i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} = 0\Pi_T + \left(q^2 - \frac{(q^2)^2}{\tilde{q}_0^2 + q^2}\right)\Pi_E = \frac{\tilde{q}_0^2 q^2}{\tilde{q}_0^2 + q^2}\Pi_E \,, \tag{8.128}$$

where L denotes the integral in Eq. (8.192). The integral on the left-hand side of Eq. (8.128) can be written as

$$\begin{aligned} \int \frac{{\rm d}\Omega_v}{4\pi} \frac{(\mathbf{q}\cdot\mathbf{v})^2 i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} &= \int \frac{{\rm d}\Omega_v}{4\pi} \frac{(-\mathbf{q}\cdot\mathbf{v} + i\tilde{q}_0 - i\tilde{q}_0)(-\mathbf{q}\cdot\mathbf{v}) i\tilde{q}_0}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \\ &= (i\tilde{q}_0)^2 \int \frac{{\rm d}\Omega_v}{4\pi} \frac{\mathbf{q}\cdot\mathbf{v}}{i\tilde{q}_0 - \mathbf{q}\cdot\mathbf{v}} \\ &= (i\tilde{q}_0)^2 \Big[-1 + i\tilde{q}_0 L\Big] \,. \end{aligned} \tag{8.129}$$

[23] V.P. Silin, *On the electromagnetic properties of a relativistic plasma,* Sov. Phys. JETP 11 (1960) 1136 [Zh. Eksp. Teor. Fiz. 38 (1960) 1577]; V.V. Klimov, *Collective Excitations in a Hot Quark Gluon Plasma,* Sov. Phys. JETP 55 (1982) 199 [Zh. Eksp. Teor. Fiz. 82 (1982) 336]; H.A. Weldon, *Covariant Calculations at Finite Temperature: the Relativistic Plasma,* Phys. Rev. D 26 (1982) 1394.

Inserting the expression from Eq. (8.192), we thus get

$$\Pi_T(\tilde{Q}) = \frac{m_{\rm E}^2}{2}\left\{\frac{(i\tilde{q}_0)^2}{\mathbf{q}^2} + \frac{i\tilde{q}_0}{2|\mathbf{q}|}\left[1-\frac{(i\tilde{q}_0)^2}{\mathbf{q}^2}\right]\ln\frac{i\tilde{q}_0+|\mathbf{q}|}{i\tilde{q}_0-|\mathbf{q}|}\right\}, \quad (8.130)$$

$$\Pi_E(\tilde{Q}) = m_{\rm E}^2\left[1-\frac{(i\tilde{q}_0)^2}{\mathbf{q}^2}\right]\left[1-\frac{i\tilde{q}_0}{2|\mathbf{q}|}\ln\frac{i\tilde{q}_0+|\mathbf{q}|}{i\tilde{q}_0-|\mathbf{q}|}\right]. \quad (8.131)$$

Eqs. (8.130), (8.131) have a number of interesting limiting values. In the limit $i\tilde{q}_0 \to 0$ but with $|\mathbf{q}| \neq 0$, $\Pi_T \to 0$, $\Pi_E \to m_{\rm E}^2$. This corresponds to the physics of *Debye screening*, familiar to us from Eq. (5.95). On the contrary, if we consider homogeneous but time-dependent waves, i.e. take $|\mathbf{q}| \to 0$ with $i\tilde{q}_0 \neq 0$, it can be seen that Π_T, $\Pi_E \to m_{\rm E}^2/3$. This genuinely Minkowskian structure in the resummed self-energy corresponds to *plasma oscillations*, or *plasmons*.

We can now also write down a general version of a resummed gluon propagator: in a general gauge, where the tree-level propagator has the form in Eq. (5.45) and the static resummed propagator the form in Eq. (5.95), we get

$$\langle A_\mu^a(\tilde{x})A_\nu^b(\tilde{y})\rangle = \delta^{ab}\,\,\,\mathclap{\displaystyle\int}\mathclap{\textstyle\sum}_{\tilde{Q}}\; e^{i\tilde{Q}\cdot(\tilde{x}-\tilde{y})}\left[\frac{P_{\mu\nu}^T(\tilde{Q})}{\tilde{Q}^2+\Pi_T(\tilde{Q})} + \frac{P_{\mu\nu}^E(\tilde{Q})}{\tilde{Q}^2+\Pi_E(\tilde{Q})} + \xi\frac{\tilde{q}_\mu\tilde{q}_\nu}{(\tilde{Q}^2)^2}\right], \quad (8.132)$$

where ξ is the gauge parameter.

If the propagator of Eq. (8.132) is used in practical computations, it is often useful to express it in terms of the *spectral representation*, cf. Exercise 11. The spectral function appearing in the spectral representation can be obtained from Eq. (8.28), where now $1/[\tilde{Q}^2+\Pi_{T(E)}(\tilde{Q})]$ plays the role of what we there denoted by $\Pi_{\alpha\beta}^E$. After analytic continuation, $i\tilde{q}_0 \to q^0 + i0^+$,

$$\frac{1}{\tilde{Q}^2+\Pi_{T(E)}(\tilde{q}_0,\tilde{\mathbf{q}})} \to \frac{1}{-(q^0+i0^+)^2+\mathbf{q}^2+\Pi_{T(E)}(-i(q^0+i0^+),\mathbf{q})}, \quad (8.133)$$

where

$$\Pi_T(-i(q^0+i0^+),\mathbf{q}) = \frac{m_{\rm E}^2}{2}\left\{\frac{(q^0)^2}{\mathbf{q}^2} + \frac{q^0}{2|\mathbf{q}|}\left[1-\frac{(q^0)^2}{\mathbf{q}^2}\right]\ln\frac{q^0+i0^++|\mathbf{q}|}{q^0+i0^+-|\mathbf{q}|}\right\}, \quad (8.134)$$

$$\Pi_E(-i(q^0+i0^+),\mathbf{q}) = m_{\rm E}^2\left[1-\frac{(q^0)^2}{\mathbf{q}^2}\right]\left[1-\frac{q^0}{2|\mathbf{q}|}\ln\frac{q^0+i0^++|\mathbf{q}|}{q^0+i0^+-|\mathbf{q}|}\right]. \quad (8.135)$$

For $|q^0| > |\mathbf{q}|$, Π_T, Π_E are real. For $|q^0| < |\mathbf{q}|$, they have an imaginary part. In particular, for $|q^0| \ll |\mathbf{q}|$, we get

$$\Pi_T \approx \frac{m_{\rm E}^2}{2}\left\{-i\pi\frac{q^0}{2|\mathbf{q}|} + 2\frac{(q^0)^2}{\mathbf{q}^2} + ...\right\}, \quad (8.136)$$

$$\Pi_E \approx m_{\rm E}^2\left\{1 + i\pi\frac{q^0}{2|\mathbf{q}|} - 2\frac{(q^0)^2}{\mathbf{q}^2} + ...\right\}. \quad (8.137)$$

The spectral functions become

$$\rho_T(q^0,\mathbf{q}) = \begin{cases} \pi\,\mathrm{sign}(q^0)\delta\Big((q^0)^2-\mathbf{q}^2-\mathrm{Re}\,\Pi_T\Big), & |q^0| > |\mathbf{q}| \\ \pi m_{\rm E}^2\dfrac{q^0}{4|\mathbf{q}|^5}, & |q^0| \ll |\mathbf{q}| \end{cases}, \quad (8.138)$$

$$\rho_E(q^0,\mathbf{q}) = \begin{cases} \pi\,\mathrm{sign}(q^0)\delta\Big((q^0)^2-\mathbf{q}^2-\mathrm{Re}\,\Pi_E\Big), & |q^0| > |\mathbf{q}| \\ -\pi m_{\rm E}^2\dfrac{q^0}{2|\mathbf{q}|(\mathbf{q}^2+m_{\rm E}^2)^2}, & |q^0| \ll |\mathbf{q}| \end{cases}. \quad (8.139)$$

In fact, introducing the notation

$$y \equiv \frac{q^0}{|\mathbf{q}|}\;, \quad q \equiv |\mathbf{q}|\;, \tag{8.140}$$

the complete expressions for ρ_T, ρ_E can be written as

$$\rho_T(q^0,\mathbf{q}) = \theta(y^2-1)\pi\,\mathrm{sign}(y)\delta(\Delta_T(y,q)) + \frac{\theta(1-y^2)\Gamma_T(y,q)}{\Delta_T^2(y,q)+\Gamma_T^2(y,q)}\;, \tag{8.141}$$

$$\Delta_T(y,q) \equiv q^2(y^2-1) - \frac{m_\mathrm{E}^2}{2}\biggl[y^2 + \frac{y}{2}\Bigl(1-y^2\Bigr)\ln\Bigl|\frac{y+1}{y-1}\Bigr|\biggr]\;, \tag{8.142}$$

$$\Gamma_T(y,q) \equiv \frac{\pi m_\mathrm{E}^2}{4}y\Bigl(1-y^2\Bigr)\;, \tag{8.143}$$

$$(y^2-1)\rho_E(q^0,\mathbf{q}) = \theta(y^2-1)\pi\,\mathrm{sign}(y)\delta(\Delta_E(y,q)) + \frac{\theta(1-y^2)\Gamma_E(y,q)}{\Delta_E^2(y,q)+\Gamma_E^2(y,q)}\;, \tag{8.144}$$

$$\Delta_E(y,q) \equiv q^2 + m_\mathrm{E}^2\biggl[1 - \frac{y}{2}\ln\Bigl|\frac{y+1}{y-1}\Bigr|\biggr]\;, \tag{8.145}$$

$$\Gamma_E(y,q) \equiv \frac{\pi m_\mathrm{E}^2}{2}y\;. \tag{8.146}$$

It can be seen that there is in each case a "plasmon" pole, i.e. a delta function analogous to the delta functions of the free propagator, Eq. (8.36), but displaced by an amount $\propto m_\mathrm{E}^2$; as well as a cut at $|y| < 1$, representing Landau damping.

So far, we have only computed the form of the resummed gluon propagator. A very interesting question is whether also an *effective action* can be written, which would then not only contain the propagators, but also new vertices, in analogy with the Dimensionally Reduced effective theory of Eq. (6.36) in the static case. Note that since our observables are now non-static, the effective action should be gauge-invariant also in time-dependent gauge transformations.

Most remarkable, such an effective action can indeed be found[24]. We simply cite here the result for the gluonic case. Expressing everything in Minkowskian notation (i.e. after $i\tilde{q}_0 \to q^0$ and using the Minkowskian A_0^a), the effective Lagrangian reads

$$\mathcal{L}_M = -\frac{1}{2}\,\mathrm{Tr}\,[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2}\,m_\mathrm{E}^2\int\frac{\mathrm{d}\Omega_v}{4\pi}\mathrm{Tr}\biggl[\Bigl(\frac{1}{v\cdot\mathcal{D}}v^\alpha F_{\alpha\mu}\Bigr)\Bigl(\frac{1}{v\cdot\mathcal{D}}v^\beta F_\beta{}^\mu\Bigr)\biggr]\;. \tag{8.147}$$

Here $v \equiv (1,\mathbf{v})$ is a light-like four-velocity, and $\mathcal{D}$ represents the covariant derivative in the adjoint representation.

Several remarks on Eq. (8.147) are in order:

- A somewhat tedious analysis, making use of the velocity integrals listed in Eqs. (8.195)–(8.203), shows that in the static limit the second term in Eq. (8.147) indeed reduces to the mass term in Eq. (6.36) (modulo Wick rotation and the redefinition of A_0^a).
- In the static limit, we found quarks to always be infrared-safe, but this situation changes after the analytic continuation. Therefore a "dynamical" quark part should be added to Eq. (8.147)[24].
- Eq. (8.147) has the unpleasant feature that it is *non-local*: derivatives appear in the denominator. This is something we do not usually expect from effective theories. Indeed, if non-local structures can appear, it is quite difficult to generally analyse what kind of higher

[24] J. Frenkel and J.C. Taylor, *Hard thermal QCD, forward scattering and effective actions,* Nucl. Phys. B 374 (1992) 156; E. Braaten and R.D. Pisarski, *Simple effective Lagrangian for hard thermal loops,* Phys. Rev. D 45 (1992) 1827.

order operators have been truncated from the effective theory and, consequently, what the relative accuracy of the effective description is.

In some sense, the appearance of non-local terms is a manifestation of the fact that the proper infrared degrees of freedom have not been identified. In the next section, we discuss a reformulation of the HTL effective theory which has additional degrees of freedom, and consequently a local appearance, thereby perhaps curing the problems. (The reformulation does not have the form of a field theory, however, whereby it continues to be difficult to analyse the accuracy of the effective description.)

- We arrived at Eq. (8.147) by integrating out the hard modes, with momenta $s \sim T$. However, like in the static limit, the theory still has multiple dynamical momentum scales, $|\mathbf{q}| \sim gT, g^2T$. It can be asked what happens if the momenta $|\mathbf{q}| \sim gT$ are also integrated out. This question has been analysed in detail in the literature, and leads indeed to a simplified (local) effective description[25], which can be used for non-perturbatively studying observables sensitive to "ultrasoft" momenta, $|\mathbf{q}| \sim g^2T$.

- Apart from observables sensitive to ultrasoft momenta, another situation where loop corrections need to be computed within the theory of Eq. (8.147) is the physics around the plasmon poles, i.e. the δ-functions in Eqs. (8.141), (8.144). Indeed, it turns out that the plasmons get a finite "width" through loop corrections, whereby the delta function gets "smeared" for $|q^0 - m_\mathrm{E}/\sqrt{3}| \lesssim g^2T$.[26]

[25] D. Bödeker, *On the effective dynamics of soft non-abelian gauge fields at finite temperature,* Phys. Lett. B 426 (1998) 351; P. Arnold, D.T. Son and L.G. Yaffe, *Hot B violation, color conductivity, and* $\log(1/\alpha)$ *effects,* Phys. Rev. D 59 (1999) 105020; D. Bödeker, *Diagrammatic approach to soft non-Abelian dynamics at high temperature,* Nucl. Phys. B 566 (2000) 402; *From hard thermal loops to Langevin dynamics,* Nucl. Phys. B 559 (1999) 502; P. Arnold and L.G. Yaffe, *High temperature color conductivity at next-to-leading log order,* Phys. Rev. D 62 (2000) 125014; D. Bödeker, *Perturbative and non-perturbative aspects of the non-abelian Boltzmann-Langevin equation,* Nucl. Phys. B 647 (2002) 512.

[26] E. Braaten and R.D. Pisarski, *Calculation of the gluon damping rate in hot QCD,* Phys. Rev. D 42 (1990) 2156.