

## 8.2. From Euclidean correlator to spectral function

As an application of the relations given in Sec. 8.1, let us carry out an explicit computation illustrating the steps. This computation will also find a practical application in Sec. 9.5.

To motivate the correlator, consider a model describing a right-handed neutrino ( $\tilde{N}$ ) interacting with the usual left-handed leptons ( $\bar{L}_\alpha$ ,  $\alpha = 1, 2, 3$ ) and the Higgs field ( $\tilde{\phi} = i\tau_2\phi^*$ ) through Yukawa interactions, with dimensionless but possibly complex coupling constants  $h_\alpha$ . The Minkowskian Lagrangian can be written as

$$\mathcal{L}_M = \frac{1}{2} \tilde{N} i \not{\partial} \tilde{N} - \frac{1}{2} M \tilde{N} \tilde{N} - h_\alpha \bar{L}_\alpha \tilde{\phi} a_R \tilde{N} - h_\alpha^* \tilde{N} \tilde{\phi}^\dagger a_L L_\alpha, \quad (8.69)$$

where repeated indices are summed over, and  $a_L \equiv (1 - \gamma_5)/2$ ,  $a_R \equiv (1 + \gamma_5)/2$  are chiral projectors.

Let us now consider a correlator that would play the role of a self-energy correction for the right-handed neutrino,

$$\Pi_{\alpha\beta}^E(\tilde{Q}) \equiv \int_0^\beta d\tau \int_{\mathbf{x}} e^{i\tilde{Q}\cdot\tilde{x}} a_L \langle \tilde{\phi}^\dagger(\tilde{x}) L_\alpha(\tilde{x}) \bar{L}_\beta(0) \tilde{\phi}(0) \rangle a_R. \quad (8.70)$$

Here  $\tilde{Q}$  is *fermionic*. In the Standard Model, the Higgs and lepton doublets have the forms

$$\tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_0 + i\phi_3 \\ -\phi_2 + i\phi_1 \end{pmatrix}, \quad L_\alpha = \begin{pmatrix} \nu_\alpha \\ \ell_\alpha \end{pmatrix}. \quad (8.71)$$

Carrying out the contractions, we can write

$$\begin{aligned} \Pi_{\alpha\beta}^E(\tilde{Q}) &= \frac{\delta_{\alpha\beta}}{2} \int_0^\beta d\tau \int_{\mathbf{x}} e^{i\tilde{Q}\cdot\tilde{x}} \sum_{c=0}^3 a_L \langle \ell_c(\tilde{x}) \bar{\ell}_c(0) \rangle_0 \langle \phi_c(\tilde{x}) \phi_c(0) \rangle_0 a_R \\ &= \frac{\delta_{\alpha\beta}}{2} \sum_{c=0}^3 \int_0^\beta d\tau \int_{\mathbf{x}} \not{\not{D}}_{\tilde{P}_f, \tilde{R}_b} e^{i(\tilde{Q} + \tilde{P} + \tilde{R})\cdot\tilde{x}} a_L \frac{-i\tilde{\not{P}} + m_{\ell_c}}{\tilde{P}^2 + m_{\ell_c}^2} \frac{1}{\tilde{R}^2 + m_{\phi_c}^2} a_R \\ &= \frac{\delta_{\alpha\beta}}{2} \sum_{c=0}^3 \int_{\mathbf{P}} T \sum_{\tilde{p}_{\text{of}}} \frac{-i\tilde{\not{P}} a_R}{\tilde{p}_0^2 + E_1^2} \frac{1}{(\tilde{p}_0 + \tilde{q}_0)^2 + E_2^2}, \end{aligned} \quad (8.72)$$

where we inserted the free scalar and fermion propagators, and denoted

$$E_1 \equiv \sqrt{m_{\ell_c}^2 + \mathbf{p}^2}, \quad E_2 \equiv \sqrt{m_{\phi_c}^2 + (\mathbf{p} + \mathbf{q})^2}. \quad (8.73)$$

Moreover the left and right projectors removed the mass term from the numerator. We have been somewhat implicit about the assignments of the masses  $m_{\ell_c}$ ,  $m_{\phi_c}$  to the corresponding fields, but for our purposes more details are not needed.

The essential issue in handling Eq. (8.72) is the treatment of the Matsubara sum. More generally, let us inspect the structure

$$\mathcal{F} = T \sum_{\tilde{p}_{\text{of}}} \frac{f(i\tilde{p}_0, \mathbf{p}, i\tilde{q}_0, \mathbf{q})}{[\tilde{p}_0^2 + E_1^2][(\tilde{p}_0 + \tilde{q}_0)^2 + E_2^2]}, \quad (8.74)$$

where we assume that the function  $f$  in the numerator depends on its arguments *at most linearly*. We can write:

$$\begin{aligned} \mathcal{F} &= T \sum_{\tilde{p}_{\text{of}}} T \sum_{\tilde{r}_{\text{ob}}} \beta \delta(\tilde{r}_{\text{ob}} - \tilde{p}_{\text{of}} - \tilde{q}_{\text{of}}) \frac{f(i\tilde{p}_0, \mathbf{p}, i\tilde{q}_0, \mathbf{q})}{[\tilde{p}_0^2 + E_1^2][\tilde{r}_0^2 + E_2^2]} \\ &= \int_0^\beta d\tau e^{-i\tilde{q}_{\text{of}}\tau} \left\{ T \sum_{\tilde{p}_{\text{of}}} e^{-i\tilde{p}_{\text{of}}\tau} \frac{f(i\tilde{p}_0, \mathbf{p}, i\tilde{q}_0, \mathbf{q})}{\tilde{p}_0^2 + E_1^2} \right\} \left\{ T \sum_{\tilde{r}_{\text{ob}}} \frac{e^{i\tilde{r}_{\text{ob}}\tau}}{\tilde{r}_0^2 + E_2^2} \right\}, \end{aligned} \quad (8.75)$$

where we wrote

$$\beta\delta(\tilde{r}_{0b} - \tilde{p}_{0f} - \tilde{q}_{0f}) = \int_0^\beta d\tau e^{i(\tilde{r}_{0b} - \tilde{p}_{0f} - \tilde{q}_{0f})\tau}. \quad (8.76)$$

Now we can make use of Eqs. (8.30), (8.63) and time derivatives thereof:

$$T \sum_{\tilde{r}_{0b}} \frac{e^{i\tilde{r}_{0b}\tau}}{\tilde{r}_0^2 + E_2^2} = \frac{n_B(E_2)}{2E_2} \left[ e^{(\beta-\tau)E_2} + e^{\tau E_2} \right], \quad (8.77)$$

$$T \sum_{\tilde{p}_{0f}} \frac{e^{\pm i\tilde{p}_{0f}\tau}}{\tilde{p}_0^2 + E_1^2} = \frac{n_F(E_1)}{2E_1} \left[ e^{(\beta-\tau)E_1} - e^{\tau E_1} \right], \quad (8.78)$$

$$T \sum_{\tilde{p}_{0f}} \frac{i\tilde{p}_0 e^{-i\tilde{p}_0\tau}}{\tilde{p}_0^2 + E_1^2} = \frac{n_F(E_1)}{2E_1} \left[ E_1 e^{(\beta-\tau)E_1} + E_1 e^{\tau E_1} \right]. \quad (8.79)$$

Thereby

$$\begin{aligned} \mathcal{F} = \int_0^\beta d\tau e^{-i\tilde{q}_{0f}\tau} \frac{n_F(E_1)n_B(E_2)}{4E_1E_2} \times \\ \times \left\{ \begin{aligned} & e^{(\beta-\tau)(E_1+E_2)} f(E_1, \mathbf{p}, i\tilde{q}_0, \mathbf{q}) \\ & + e^{(\beta-\tau)E_2+\tau E_1} f(E_1, -\mathbf{p}, -i\tilde{q}_0, -\mathbf{q}) \\ & + e^{(\beta-\tau)E_1+\tau E_2} f(E_1, \mathbf{p}, i\tilde{q}_0, \mathbf{q}) \\ & + e^{\tau(E_1+E_2)} f(E_1, -\mathbf{p}, -i\tilde{q}_0, -\mathbf{q}) \end{aligned} \right\}. \quad (8.80) \end{aligned}$$

As an example, let us focus on the third structure in Eq. (8.80). The  $\tau$ -integral can be carried out, noting that  $\tilde{q}_{0f}$  is fermionic:

$$\begin{aligned} \int_0^\beta d\tau e^{\beta E_1} e^{\tau(-i\tilde{q}_{0f}-E_1+E_2)} &= \frac{e^{\beta E_1}}{-i\tilde{q}_0 - E_1 + E_2} \left[ -e^{\beta(E_2-E_1)} - 1 \right] \\ &= \frac{e^{\beta E_2} + e^{\beta E_1}}{i\tilde{q}_0 + E_1 - E_2} \\ &= \frac{1}{i\tilde{q}_0 + E_1 - E_2} \left[ n_B^{-1}(E_2) + n_F^{-1}(E_1) \right]. \quad (8.81) \end{aligned}$$

Thus

$$\mathcal{F}|_{\text{3rd}} = \frac{1}{4E_1E_2} \left[ n_F(E_1) + n_B(E_2) \right] \frac{f(E_1, \mathbf{p}, i\tilde{q}_0, \mathbf{q})}{i\tilde{q}_0 + E_1 - E_2}. \quad (8.82)$$

Finally we set  $\tilde{q}_0 \rightarrow -i(q^0 + i0^+)$  and take the imaginary part according to Eq. (8.28). Furthermore, making use of Eq. (8.21), we note that

$$\frac{1}{2i} \left[ \frac{1}{q^0 + \Delta + i0^+} - \frac{1}{q^0 + \Delta - i0^+} \right] = -\pi\delta(q^0 + \Delta), \quad (8.83)$$

whereby the denominator in Eq. (8.82) simply gets replaced with  $(-\pi)$  times a Dirac delta function. Special attention needs to be paid here to the possibility that  $\tilde{q}_0$  could also appear in the numerator in Eq. (8.82); however, we can then write

$$i\tilde{q}_0 = \underbrace{i\tilde{q}_0 + E_1 - E_2}_{\text{no discontinuity}} + E_2 - E_1, \quad (8.84)$$



$$\begin{aligned}
&= \delta(E_1 - E_2 + q^0) \frac{(e^{-\beta q^0} + 1)e^{\beta E_2}}{(e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\
&= \delta(E_1 - E_2 + q^0)(e^{\beta q^0} + 1) \frac{e^{\beta(E_2 - q^0)}}{(e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\
&= \delta(E_1 - E_2 + q^0) n_{\text{F}}^{-1}(q^0) \frac{e^{\beta E_1}}{(e^{\beta E_1} + 1)(e^{\beta E_2} - 1)} \\
&= n_{\text{F}}^{-1}(q^0) \delta(E_2 - E_1 - q^0) n_{\text{B}2} (1 - n_{\text{F}1}) . \quad (8.90)
\end{aligned}$$

We see that we get back the structure of the 2nd channel, which indeed is the desired result, because that is precisely what we need from the point of view of changing  $\mathbf{q} \rightarrow -\mathbf{q}$  in the delta function conserving the spatial momentum. Similarly, it can be checked that the 4th term goes over into the 1st term, and vica versa.

### 8.3. Exercise 11

In the text we made use of tree-level propagators, but in general the propagators need to be resummed, and obtain a more complicated form. In this situation it may be useful to express them as in the spectral representation of Eq. (8.25). In particular, the scalar propagator can in general be written as

$$\langle \tilde{\phi}(\tilde{P})\tilde{\phi}(\tilde{Q}) \rangle_0 = \frac{\delta(\tilde{P} + \tilde{Q})}{\tilde{q}_0^2 + \mathbf{q}^2 + \Pi_S(\tilde{q}_0, \mathbf{q})} = \delta(\tilde{P} + \tilde{Q}) \int_{-\infty}^{\infty} \frac{dq^0}{\pi} \frac{\rho_S(q^0, \mathbf{q})}{q^0 - i\tilde{q}_0}, \quad (8.91)$$

while the fermion propagator contains two possible structures (or, if chirality is not a symmetry, even more)<sup>21</sup>,

$$\begin{aligned} \langle \tilde{\psi}(\tilde{P})\tilde{\psi}(\tilde{Q}) \rangle_0 &= \delta(\tilde{P} - \tilde{Q}) \left[ \frac{-i\tilde{Q}}{\tilde{q}_0^2 + \mathbf{q}^2 + \Pi_L(\tilde{q}_0, \mathbf{q})} + \frac{\tilde{\gamma}_0}{\tilde{q}_0^2 + \mathbf{q}^2 + \Pi_U(\tilde{q}_0, \mathbf{q})} \right] \\ &= \delta(\tilde{P} - \tilde{Q}) \left[ -i\tilde{Q} \int_{-\infty}^{\infty} \frac{dq^0}{\pi} \frac{\rho_L(q^0, \mathbf{q})}{q^0 - i\tilde{q}_0} + \not{u} \int_{-\infty}^{\infty} \frac{dq^0}{\pi} \frac{\rho_U(q^0, \mathbf{q})}{q^0 - i\tilde{q}_0} \right], \end{aligned} \quad (8.92)$$

where  $u = (1, \mathbf{0})$  is the plasma four-velocity. Carry out the steps from Eq. (8.72) to (8.88) in this situation.

### Solution to Exercise 11

The structure in Eq. (8.74) now becomes

$$\mathcal{F} = T \sum_{\tilde{p}_{0f}} \sum_{F=L,U} \int_{-\infty}^{\infty} \frac{d\omega_1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{\pi} \frac{f_F(i\tilde{p}_0, \mathbf{p}, u) \rho_F(\omega_1, \mathbf{p}) \rho_S(\omega_2, \mathbf{p} + \mathbf{q})}{[\omega_1 - i\tilde{p}_0][\omega_2 - i(\tilde{p}_0 + \tilde{q}_0)]}, \quad (8.93)$$

where we assume that the function  $f_F$  in the numerator depends on its arguments *at most linearly*. We can write:

$$\begin{aligned} \mathcal{F} &= \sum_{F=L,U} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{\pi^2} \rho_F(\omega_1, \mathbf{p}) \rho_S(\omega_2, \mathbf{p} + \mathbf{q}) \times \\ &\times T \sum_{\tilde{p}_{0f}} T \sum_{\tilde{r}_{0b}} \beta \delta(\tilde{r}_{0b} - \tilde{p}_{0f} - \tilde{q}_{0f}) \frac{f_F(i\tilde{p}_0, \mathbf{p}, u)}{[\omega_1 - i\tilde{p}_0][\omega_2 - i\tilde{r}_0]}. \end{aligned} \quad (8.94)$$

Employing Eqs. (8.76), (8.31) and (8.62), as well as the time derivative of the last one,

$$T \sum_{\omega_f} \frac{i\omega_f}{E - i\omega_f} e^{-i\omega_f \tau} = -\frac{d}{d\tau} \left[ n_F(E) e^{(\beta - \tau)E} \right] = n_F(E) E e^{(\beta - \tau)E}, \quad (8.95)$$

we get

$$\begin{aligned} \mathcal{F} &= \sum_{F=L,U} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{\pi^2} \rho_F(\omega_1, \mathbf{p}) \rho_S(\omega_2, \mathbf{p} + \mathbf{q}) \times \\ &\times \int_0^\beta d\tau e^{-i\tilde{q}_{0f}\tau} n_F(\omega_1) n_B(\omega_2) f_F(\omega_1, \mathbf{p}, u) e^{(\beta - \tau)\omega_1 + \tau\omega_2}. \end{aligned} \quad (8.96)$$

<sup>21</sup>H.A. Weldon, *Effective Fermion Masses of  $\mathcal{O}(gT)$  in High Temperature Gauge Theories with Exact Chiral Invariance*, Phys. Rev. D 26 (1982) 2789.

The  $\tau$ -integral can be carried out, noting that  $\tilde{q}_{0f}$  is fermionic:

$$\begin{aligned}
\int_0^\beta d\tau n_F(\omega_1)n_B(\omega_2) e^{\beta\omega_1} e^{\tau(-i\tilde{q}_{0f}-\omega_1+\omega_2)} &= \frac{n_F(\omega_1)n_B(\omega_2)e^{\beta\omega_1}}{-i\tilde{q}_0-\omega_1+\omega_2} \left[-e^{\beta(\omega_2-\omega_1)}-1\right] \\
&= \frac{n_F(\omega_1)n_B(\omega_2)}{i\tilde{q}_0+\omega_1-\omega_2} \left[e^{\beta\omega_2}+e^{\beta\omega_1}\right] \\
&= \frac{n_F(\omega_1)n_B(\omega_2)}{i\tilde{q}_0+\omega_1-\omega_2} \left[n_F^{-1}(\omega_1)+n_B^{-1}(\omega_2)\right] \\
&= \frac{1}{i\tilde{q}_0+\omega_1-\omega_2} \left[n_F(\omega_1)+n_B(\omega_2)\right]. \quad (8.97)
\end{aligned}$$

Finally we set  $\tilde{q}_0 \rightarrow -i(q^0 + i0^+)$  and take the imaginary part. Making use of Eq. (8.83), the denominator in Eq. (8.97) simply gets replaced with  $(-\pi)$  times a Dirac delta function. Thus, in total,

$$\begin{aligned}
\text{Im}\left\{\mathcal{F}(i\tilde{q}_0 \rightarrow q^0 + i0^+)\right\} &= -\pi \sum_{F=L,U} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{\pi^2} \rho_F(\omega_1, \mathbf{p}) \rho_S(\omega_2, \mathbf{p} + \mathbf{q}) \times \\
&\quad \times \left[n_F(\omega_1) + n_B(\omega_2)\right] \delta(q^0 + \omega_1 - \omega_2) f_F(\omega_1, \mathbf{p}, u) \\
&= -\frac{1}{2} \sum_{F=L,U} \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{\pi^2} \rho_F(\omega_1, \mathbf{p}) \rho_S(\omega_2, \mathbf{p} + \mathbf{q}) \times \\
&\quad \times 2\pi \delta(q^0 + \omega_1 - \omega_2) f_L(\omega_1, \mathbf{p}, u) n_F^{-1}(q^0) n_B(\omega_2) [1 - n_F(\omega_1)], \quad (8.98)
\end{aligned}$$

where we paralleled the steps in Eq. (8.85). Moreover, representing

$$g(\mathbf{p} + \mathbf{q}) = \int \frac{d^3\mathbf{p}_2}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(\mathbf{q} + \mathbf{p} - \mathbf{p}_2) g(\mathbf{p}_2), \quad (8.99)$$

and defining  $P_1 \equiv (\omega_1, \mathbf{p}) \equiv (\omega_1, \mathbf{p}_1)$ ,  $P_2 \equiv (\omega_2, \mathbf{p}_2)$ , we get

$$\begin{aligned}
\rho_{\alpha\beta}(Q) &= \frac{\delta_{\alpha\beta}}{4} \sum_{c=0}^3 \frac{n_F^{-1}(q^0)}{\pi^2} \int \frac{d\omega_1 d^3\mathbf{p}_1}{(2\pi)^3} \int \frac{d\omega_2 d^3\mathbf{p}_2}{(2\pi)^3} \left[\mathcal{P}_1 \rho_L(P_1) - \not{u} \rho_U(P_1)\right] a_R \rho_S(P_2) \times \\
&\quad \times \left\{ (2\pi)^4 \delta^{(4)}(P_2 - P_1 - Q) n_{B2} (1 - n_{F1}) \right\} \quad (8.100)
\end{aligned}$$

where  $n_{Fi} \equiv n_F(\omega_i)$ ,  $n_{Bi} \equiv n_B(\omega_i)$ . If we insert the free spectral functions from Eqs. (8.36), (8.66), and note that in free limit  $\rho_U = 0$ , this result goes over into Eq. (8.88).