8. Real-time observables

We now move to a new class of observables, those including both a Minkowskian time t and a temperature T. Examples are production rates from a thermal plasma of various types of weakly-interacting particles; rates of oscillation and damping of waves travelling in the plasma; as well as transport coefficients such as electric and thermal conductivity and bulk and shear viscosity. We start by developing some aspects of the general formalism needed for treating these observables, and return later on to specific applications.

8.1. Different Green's functions

Practically all the observables of interest in the following can be reduced to *two-point correlation functions* of elementary or composite operators. Let us therefore list some common definitions and relations that apply to such correlation functions[20].

We denote Minkowskian space-time coordinates by $x = (t, x^i)$ and momenta by $Q = (q^0, q^i)$, while their Euclidean counterparts are denoted by $\tilde{x} = (\tau, x^i)$, $\tilde{Q} = (\tilde{q}_0, q_i)$. Wick rotation is carried out by $\tau \leftrightarrow it$, $\tilde{q}_0 \leftrightarrow -iq^0$. Scalar products are defined as $Q \cdot x = q^0 t + q^i x_i = q^0 t - \mathbf{q} \cdot \mathbf{x}$, $\tilde{Q} \cdot \tilde{x} = \tilde{q}_0 \tau + q_i x^i = \tilde{q}_0 \tau - \mathbf{q} \cdot \mathbf{x}$. Arguments of operators denote implicitely whether we are in Minkowskian or Euclidean space-time. In particular, Heisenberg-operators are defined as

$$\hat{O}(t, \mathbf{x}) \equiv e^{i\hat{H}t} \hat{O}(0, \mathbf{x}) e^{-i\hat{H}t} \,, \quad \hat{O}(\tau, \mathbf{x}) \equiv e^{\hat{H}\tau} \hat{O}(0, \mathbf{x}) e^{-\hat{H}\tau} \,. \tag{8.1}$$

The thermal ensemble is defined by the density matrix $\hat{\rho} = \mathcal{Z}^{-1} \exp(-\beta \hat{H})$.

Bosonic case

We first consider operators that are *bosonic* in nature, i.e. commuting (modulo possible contact terms). We denote the operators which appear in the two-point functions by $\hat{\phi}_\alpha(x)$, $\hat{\phi}^\dagger_\beta(x)$.

We can define various classes of correlation functions. The "physical" correlators are defined as

$$\Pi^{>}_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \hat{\phi}_\alpha(x) \hat{\phi}^\dagger_\beta(0) \Big\rangle \,, \tag{8.2}$$

$$\Pi^{<}_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \hat{\phi}^\dagger_\beta(0) \hat{\phi}_\alpha(x) \Big\rangle \,, \tag{8.3}$$

$$\rho_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \frac{1}{2} \Big[\hat{\phi}_\alpha(x), \hat{\phi}^\dagger_\beta(0) \Big] \Big\rangle \,, \tag{8.4}$$

$$\Delta_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \frac{1}{2} \Big\{ \hat{\phi}_\alpha(x), \hat{\phi}^\dagger_\beta(0) \Big\} \Big\rangle \,, \tag{8.5}$$

where $\rho_{\alpha\beta}$ is called the *spectral function*, while the "retarded"/"advanced" correlators can be defined as

$$\Pi^{R}_{\alpha\beta}(Q) \equiv i \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \Big[\hat{\phi}_\alpha(x), \hat{\phi}^\dagger_\beta(0) \Big] \theta(t) \Big\rangle \,, \tag{8.6}$$

$$\Pi^{A}_{\alpha\beta}(Q) \equiv i \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle -\Big[\hat{\phi}_\alpha(x), \hat{\phi}^\dagger_\beta(0) \Big] \theta(-t) \Big\rangle \,. \tag{8.7}$$

[20] A.L. Fetter and J.D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971); S. Doniach and E.H. Sondheimer, *Green's Functions for Solid State Physicists* (Benjamin, Reading, 1974); J.W. Negele and H. Orland, *Quantum Many Particle Systems* (Addison-Wesley, Redwood City, 1988).

On the other hand, from the computational point of view one is often faced with "time-ordered" correlation functions,

$$\Pi^T_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}\Big\langle \hat{\phi}_\alpha(x)\hat{\phi}^\dagger_\beta(0)\theta(t) + \hat{\phi}^\dagger_\beta(0)\hat{\phi}_\alpha(x)\theta(-t)\Big\rangle \,, \tag{8.8}$$

which appear in time-dependent perturbation theory, or with the "Euclidean" correlator

$$\Pi^E_{\alpha\beta}(\tilde{Q}) \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\, e^{i\tilde{Q}\cdot\tilde{x}}\Big\langle \hat{\phi}_\alpha(\tilde{x})\hat{\phi}^\dagger_\beta(0)\Big\rangle \,, \tag{8.9}$$

which appears in non-perturbative formulations. Note that the Euclidean correlator is also time-ordered by definition, and can be computed with standard imaginary-time functional integrals in the Matsubara formalism; $\tilde{q}_0$ is a *bosonic* Matsubara frequency

It follows from Eq. (8.1) that

$$\langle \hat{\phi}_\alpha(\beta,\mathbf{x})\hat{\phi}^\dagger_\beta(0,\mathbf{0})\rangle = \frac{1}{\mathcal{Z}}\mathrm{Tr}\left[e^{-\beta\hat{H}}e^{\beta\hat{H}}\hat{\phi}_\alpha(0,\mathbf{x})e^{-\beta\hat{H}}\hat{\phi}^\dagger_\beta(0,\mathbf{0})\right] = \langle \hat{\phi}^\dagger_\beta(0,\mathbf{0})\hat{\phi}_\alpha(0,\mathbf{x})\rangle \,. \tag{8.10}$$

This is a version of the so-called Kubo-Martin-Schwinger (KMS) relation, which in general relates $\Pi^>_{\alpha\beta}$ and $\Pi^<_{\alpha\beta}$ to each other.

More generally, *all* of the correlation functions defined above can be related to each other. In particular, all correlators can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. In order to do this, we may first insert sets of energy eigenstates into the definitions of $\Pi^>_{\alpha\beta}$ and $\Pi^<_{\alpha\beta}$:

$$\begin{aligned}
\Pi^>_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}}\int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}\mathrm{Tr}\Big[e^{-\beta\hat{H}+i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|}\hat{\phi}_\alpha(0,\mathbf{x})e^{-i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|}\hat{\phi}^\dagger_\beta(0,\mathbf{0})\Big] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}e^{(-\beta+it)E_m}e^{-itE_n}\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= \frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}}\sum_{m,n} e^{-\beta E_m}\, 2\pi\,\delta(q^0+E_m-E_n)\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \,, \qquad (8.11) \\
\Pi^<_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}}\int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}\mathrm{Tr}\Big[e^{-\beta\hat{H}}\underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|}\hat{\phi}^\dagger_\beta(0,\mathbf{0})e^{i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|}\hat{\phi}_\alpha(0,\mathbf{x})e^{-i\hat{H}t}\Big] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x}e^{(-\beta-it)E_n}e^{itE_m}\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \qquad (8.12) \\
&= \frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}}\sum_{m,n} e^{-\beta E_n}\, 2\pi\,\underbrace{\delta(q^0+E_m-E_n)}_{E_n=E_m+q^0}\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= e^{-\beta q^0}\,\Pi^>_{\alpha\beta}(Q) \,. \qquad (8.13)
\end{aligned}$$

This is the Fourier-space version of the KMS relation. Consequently

$$\rho_{\alpha\beta}(Q) = \frac{1}{2}\left[\Pi^>_{\alpha\beta}(Q) - \Pi^<_{\alpha\beta}(Q)\right] = \frac{1}{2}\left(e^{\beta q^0}-1\right)\Pi^<_{\alpha\beta}(Q) \tag{8.14}$$

and, conversely,

$$\begin{aligned}
\Pi^<_{\alpha\beta}(Q) &= 2n_{\mathrm{B}}(q^0)\rho_{\alpha\beta}(Q) \,, \qquad (8.15) \\
\Pi^>_{\alpha\beta}(Q) &= 2\frac{e^{\beta q^0}}{e^{\beta q^0}-1}\rho_{\alpha\beta}(Q) = 2[1+n_{\mathrm{B}}(q^0)]\rho_{\alpha\beta}(Q) \,, \qquad (8.16)
\end{aligned}$$

where $n_{\rm B}(x) \equiv 1/[\exp(\beta x) - 1]$. Moreover,

$$\Delta_{\alpha\beta}(Q) = \frac{1}{2}\left[\Pi^{>}_{\alpha\beta}(Q) + \Pi^{<}_{\alpha\beta}(Q)\right] = [1 + 2n_{\rm B}(q^0)]\rho_{\alpha\beta}(Q) \; . \tag{8.17}$$

Note that $1 + 2n_{\rm B}(-q^0) = -[1 + 2n_{\rm B}(q^0)]$, so that if ρ is odd in $Q \to -Q$, then Δ is even.

Inserting the representation

$$\theta(t) = i\int_{-\infty}^{\infty}\frac{{\rm d}\omega}{2\pi}\frac{e^{-i\omega t}}{\omega + i0^+} \tag{8.18}$$

into the definitions of Π^R, Π^A, we obtain

$$\begin{aligned}
\Pi^R_{\alpha\beta}(Q) &= i\int {\rm d}t\,{\rm d}^3\mathbf{x}\, e^{iQ\cdot x}\; 2\theta(t)\int\frac{{\rm d}^4P}{(2\pi)^4}\, e^{-iP\cdot x}\rho_{\alpha\beta}(P) \\
&= -2\int {\rm d}t\int\frac{{\rm d}\omega}{2\pi}\int\frac{{\rm d}p^0}{2\pi}\frac{e^{i(q^0-p^0-\omega)t}}{\omega + i0^+}\rho_{\alpha\beta}(p^0,\mathbf{q}) \\
&= -2\int\frac{{\rm d}\omega}{2\pi}\int\frac{{\rm d}p^0}{2\pi}\frac{2\pi\delta(q^0-p^0-\omega)}{\omega + i0^+}\rho_{\alpha\beta}(p^0,\mathbf{q}) \\
&= \int_{-\infty}^{\infty}\frac{{\rm d}p^0}{\pi}\frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{p^0 - q^0 - i0^+} \; ,
\end{aligned} \tag{8.19}$$

and similarly

$$\Pi^A_{\alpha\beta}(Q) = \int_{-\infty}^{\infty}\frac{{\rm d}p^0}{\pi}\frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{p^0 - q^0 + i0^+} \; . \tag{8.20}$$

Making use of

$$\frac{1}{\Delta \pm i0^+} = P\Big(\frac{1}{\Delta}\Big) \mp i\pi\delta(\Delta) \; , \tag{8.21}$$

we find

$${\rm Im}\,\Pi^R_{\alpha\beta}(Q) = \rho_{\alpha\beta}(Q) \; , \quad {\rm Im}\,\Pi^A_{\alpha\beta}(Q) = -\rho_{\alpha\beta}(Q) \; . \tag{8.22}$$

Furthermore, the real parts of Π^R and Π^A agree, so that $-i[\Pi^R_{\alpha\beta}(Q) - \Pi^A_{\alpha\beta}(Q)] = 2\rho_{\alpha\beta}(Q)$.

Moving on to Π^T, and making use of the inverse transforms of Eqs. (8.15), (8.16), we find

$$\begin{aligned}
\Pi^T_{\alpha\beta}(Q) &= \int {\rm d}t\,{\rm d}^3\mathbf{x}\, e^{iQ\cdot x}\int\frac{{\rm d}^4P}{(2\pi)^4}\, e^{-iP\cdot x}\Big[\theta(t)2e^{\beta p^0}n_{\rm B}(p^0) + \theta(-t)2n_{\rm B}(p^0)\Big]\rho_{\alpha\beta}(P) \\
&= 2i\int {\rm d}t\int\frac{{\rm d}\omega}{2\pi}\int\frac{{\rm d}p^0}{2\pi}\left[\frac{e^{i(q^0-p^0-\omega)t}}{\omega + i0^+}e^{\beta p^0} + \frac{e^{i(q^0-p^0+\omega)t}}{\omega + i0^+}\right]n_{\rm B}(p^0)\rho_{\alpha\beta}(p^0,\mathbf{q}) \\
&= 2i\int\frac{{\rm d}\omega}{2\pi}\int\frac{{\rm d}p^0}{2\pi}\left[\frac{2\pi\delta(q^0-p^0-\omega)}{\omega + i0^+}e^{\beta p^0} + \frac{2\pi\delta(q^0-p^0+\omega)}{\omega + i0^+}\right]n_{\rm B}(p^0)\rho_{\alpha\beta}(p^0,\mathbf{q}) \\
&= i\int\frac{{\rm d}p^0}{\pi}\left[\frac{e^{\beta p^0}}{q^0 - p^0 + i0^+} - \frac{1}{q^0 - p^0 - i0^+}\right]n_{\rm B}(p^0)\rho_{\alpha\beta}(p^0,\mathbf{q}) \\
&= \int_{-\infty}^{\infty}\frac{{\rm d}p^0}{\pi}\frac{i\rho_{\alpha\beta}(p^0,\mathbf{q})}{q^0 - p^0 + i0^+} + 2\rho_{\alpha\beta}(q^0,\mathbf{q})n_{\rm B}(q^0) \\
&= -i\Pi^R_{\alpha\beta}(Q) + \Pi^<_{\alpha\beta}(Q) \; ,
\end{aligned} \tag{8.23}$$

where in the penultimate step we inserted Eq. (8.21) as well as the identity $n_{\rm B}(p^0)e^{\beta p^0} = 1 + n_{\rm B}(p^0)$.

Finally, we note that the sums in Eq. (8.11) are exponentially convergent for $0 < it < \beta$. Therefore we can relate the two functions

$$\Big\langle\hat\phi_\alpha(x)\hat\phi^\dagger_\beta(0)\Big\rangle \quad \text{and} \quad \Big\langle\hat\phi_\alpha(\tilde x)\hat\phi^\dagger_\beta(0)\Big\rangle \tag{8.24}$$

by a direct analytic continuation $t \to -i\tau$, ot $it \to \tau$, with $0 < \tau < \beta$. Thereby

$$\begin{aligned}
\Pi^E_{\alpha\beta}(\tilde{Q}) &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\, e^{i\tilde{Q}\cdot\tilde{x}} \left[\int \frac{\mathrm{d}^4P}{(2\pi)^4}\, e^{-iP\cdot x} \Pi^>_{\alpha\beta}(p^0,\mathbf{p}) \right]_{it\to\tau} \\
&= \int_0^\beta \mathrm{d}\tau\, e^{i\tilde{q}_0\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} e^{-p^0\tau} \Pi^>_{\alpha\beta}(p^0,\mathbf{q}) \\
&= \int_0^\beta \mathrm{d}\tau\, e^{i\tilde{q}_0\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} e^{-p^0\tau} \frac{2e^{\beta p^0}}{e^{\beta p^0}-1} \rho_{\alpha\beta}(p^0,\mathbf{q}) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{\pi} \frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{1-e^{-\beta p^0}} \left[\frac{e^{(i\tilde{q}_0-p^0)\tau}}{i\tilde{q}_0-p^0} \right]_0^\beta \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{\pi} \frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{1-e^{-\beta p^0}} \frac{e^{-\beta p^0}-1}{i\tilde{q}_0-p^0} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{\pi} \frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{p^0-i\tilde{q}_0}\,, \qquad (8.25)
\end{aligned}$$

where we inserted Eq. (8.16) for $\Pi^>(Q)$, and changed orders of integration. This relation is called the *spectral representation* of the Euclidean correlator.

It is useful to note that Eq. (8.25) implies the existence of a simple "sum rule":

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}q^0}{\pi} \frac{\rho_{\alpha\beta}(q^0,\mathbf{q})}{q^0} = \int_0^\beta \mathrm{d}\tau\, \Pi^E_{\alpha\beta}(\tau,\mathbf{q})\,. \qquad (8.26)$$

Here we set $\tilde{q}_0 = 0$ and used the definition in Eq. (8.9) on the left-hand side of Eq. (8.25). The usefulness of the sum rule is that it relates directly (integrals over) Minkowskian and Euclidean correlators to each other.

Finally, the spectral representation can formally be inverted by making use of Eq. (8.21),

$$\begin{aligned}
\rho_{\alpha\beta}(q^0,\mathbf{q}) &= \frac{1}{2i} \,\mathrm{Disc}\, \Pi^E_{\alpha\beta}(\tilde{q}_0 \to -iq^0,\mathbf{q}) \qquad (8.27) \\
&\equiv \frac{1}{2i} \Big[\Pi^E_{\alpha\beta}(-i[q^0+i0^+],\mathbf{q}) - \Pi^E_{\alpha\beta}(-i[q^0-i0^+],\mathbf{q}) \Big]\,. \qquad (8.28)
\end{aligned}$$

Furthermore, a comparison of Eqs. (8.19) and (8.25) shows that

$$\Pi^R_{\alpha\beta}(Q) = \Pi^E_{\alpha\beta}(\tilde{q}_0 \to -i[q^0+i0^+],\mathbf{q})\,. \qquad (8.29)$$

In the context of the spectral representation, Eq. (8.25), it will often be useful to note from Eq. (4.75), viz.

$$T\sum_{\omega_\mathrm{b}} \frac{e^{i\omega_\mathrm{b}\tau}}{\omega_\mathrm{b}^2+\omega^2} = \frac{n_\mathrm{B}(\omega)}{2\omega} \Big[e^{(\beta-\tau)\omega} + e^{\tau\omega} \Big]\,, \qquad (8.30)$$

that, for $0 < \tau < \beta$,

$$\begin{aligned}
T\sum_{\omega_\mathrm{b}} \frac{1}{p^0-i\omega_\mathrm{b}} e^{i\omega_\mathrm{b}\tau} &= T\sum_{\omega_\mathrm{b}} \frac{i\omega_\mathrm{b}+p^0}{\omega_\mathrm{b}^2+(p^0)^2} e^{i\omega_\mathrm{b}\tau} \\
&= (\partial_\tau + p^0) T \sum_{\omega_\mathrm{b}} \frac{e^{i\omega_\mathrm{b}\tau}}{\omega_\mathrm{b}^2+(p^0)^2} \\
&= \frac{n_\mathrm{B}(p^0)}{2p^0} \Big[(-p^0+p^0) e^{(\beta-\tau)p^0} + (p^0+p^0) e^{\tau p^0} \Big] \\
&= n_\mathrm{B}(p^0) e^{\tau p^0}\,. \qquad (8.31)
\end{aligned}$$

This relation turns out to be valid both for both $p^0 < 0$ and $p^0 > 0$ (to show this, substitute $\omega_{\rm b} \to -\omega_{\rm b}$ and use Eq. (8.32)). We also note that, again for $0 < \tau < \beta$,

$$T\sum_{\omega_{\rm b}} \frac{1}{p^0 - i\omega_{\rm b}} e^{-i\omega_{\rm b}\tau} \;=\; T\sum_{\omega_{\rm b}} \frac{1}{p^0 - i\omega_{\rm b}} e^{i\omega_{\rm b}(\beta-\tau)} = n_{\rm B}(p^0) e^{(\beta-\tau)p^0} \,. \tag{8.32}$$

In particular, taking the inverse Fourier transform ($T\sum_{\tilde{q}_0} e^{-i\tilde{q}_0\tau}$) from the left-hand side of Eq. (8.25), and employing Eq. (8.32), we get the relation

$$\begin{aligned}
&\int \mathrm{d}^3\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \Big\langle \hat{\phi}_\alpha(\tau,\mathbf{x}) \hat{\phi}^\dagger_\beta(0,\mathbf{0}) \Big\rangle \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{\pi} \rho_{\alpha\beta}(p^0,\mathbf{q}) n_{\rm B}(p^0) e^{(\beta-\tau)p^0} \\
&= \int_0^{\infty} \frac{\mathrm{d}p^0}{\pi} \left\{ \frac{\rho_{\alpha\beta}(p^0) + \rho_{\alpha\beta}(-p^0)}{2} \frac{\sinh\Big[\Big(\frac{\beta}{2} - \tau\Big)p^0\Big]}{\sinh\Big(\frac{\beta}{2}p^0\Big)} + \frac{\rho_{\alpha\beta}(p^0) - \rho_{\alpha\beta}(-p^0)}{2} \frac{\cosh\Big[\Big(\frac{\beta}{2} - \tau\Big)p^0\Big]}{\sinh\Big(\frac{\beta}{2}p^0\Big)} \right\} ,
\end{aligned} \tag{8.33}$$

where we symmetrised and anti-symmetrised the "kernel" $n_{\rm B}(p^0)e^{(\beta-\tau)p^0}$ with respect to p^0, and for brevity left out the argument $\mathbf{q}$ from $\rho_{\alpha\beta}$ on the last line. Normally (when $\hat{\phi}_\alpha$ and $\hat{\phi}^\dagger_\beta$ are identical) the spectral function is antisymmetric in $p^0 \to -p^0$, and then only the second term on the last line of Eq. (8.33) contributes. Thereby we obtain a potentially very powerful identity: if the left-hand side of Eq. (8.33) can be measured non-perturbatively on a Euclidean lattice with Monte Carlo simulations as a function of τ, then an "inversion" of Eq. (8.33) could lead to a non-perturbative estimate of the Minkowskian spectral function.

Example

Let us illustrate the use of some of the relations obtained above with the example of a free propagator in scalar field theory:

$$\begin{aligned}
\Pi^E(Q) \;&=\; \frac{1}{\tilde{q}_0^2 + E_{\mathbf{q}}^2} \\
&=\; \frac{1}{2E_{\mathbf{q}}} \bigg(\frac{1}{i\tilde{q}_0 + E_{\mathbf{q}}} + \frac{1}{-i\tilde{q}_0 + E_{\mathbf{q}}} \bigg) \,,
\end{aligned} \tag{8.34}$$

where $E_{\mathbf{q}} = \sqrt{m^2 + \mathbf{q}^2}$. According to Eqs. (8.29), (8.21)

$$\begin{aligned}
\Pi^R(Q) \;&=\; \frac{1}{2E_{\mathbf{q}}} \bigg(\frac{1}{q^0 + E_{\mathbf{q}} + i0^+} + \frac{1}{-q^0 + E_{\mathbf{q}} - i0^+} \bigg) \\
&=\; \frac{1}{2E_{\mathbf{q}}} \Big\{ P\Big(\frac{1}{q^0 + E_{\mathbf{q}}}\Big) - P\Big(\frac{1}{q^0 - E_{\mathbf{q}}}\Big) + i\pi \Big[\delta(q^0 - E_{\mathbf{q}}) - \delta(q^0 + E_{\mathbf{q}})\Big] \Big\} \\
&=\; -P\Big(\frac{1}{(q^0)^2 - E_{\mathbf{q}}^2}\Big) + \frac{i\pi}{2E_{\mathbf{q}}} \Big[\delta(q^0 - E_{\mathbf{q}}) - \delta(q^0 + E_{\mathbf{q}})\Big] \,,
\end{aligned} \tag{8.35}$$

and according to Eq. (8.22),

$$\rho(Q) \;=\; \frac{\pi}{2E_{\mathbf{q}}} \Big[\delta(q^0 - E_{\mathbf{q}}) - \delta(q^0 + E_{\mathbf{q}})\Big] \,. \tag{8.36}$$

Finally, according to Eq. (8.23),

$$\Pi^T(Q) \;=\; P\Big(\frac{i}{(q^0)^2 - E_{\mathbf{q}}^2}\Big) + \frac{\pi}{2E_{\mathbf{q}}} \Big\{ \delta(q^0 - E_{\mathbf{q}})[1 + 2n_{\rm B}(q^0)] - \delta(q^0 + E_{\mathbf{q}})[1 + 2n_{\rm B}(q^0)] \Big\}$$

$$
\begin{aligned}
&= P\Big(\frac{i}{(q^0)^2 - E_{\mathbf{q}}^2}\Big) + \frac{\pi}{2E_{\mathbf{q}}}\Big[\delta(q^0 - E_{\mathbf{q}}) + \delta(q^0 + E_{\mathbf{q}})\Big][1 + 2n_{\mathrm{B}}(|q^0|)] \\
&= P\Big(\frac{i}{(q^0)^2 - E_{\mathbf{q}}^2}\Big) + \pi\delta\Big((q^0)^2 - E_{\mathbf{q}}^2\Big)[1 + 2n_{\mathrm{B}}(|q^0|)] \\
&= \frac{i}{(q^0)^2 - E_{\mathbf{q}}^2 + i0^+} + 2\pi\delta\Big((q^0)^2 - E_{\mathbf{q}}^2\Big)n_{\mathrm{B}}(|q^0|) \; , \qquad (8.37)
\end{aligned}
$$

where in the second step we made use of the identity $1 + 2n_{\mathrm{B}}(-E_{\mathbf{q}}) = -[1 + 2n_{\mathrm{B}}(E_{\mathbf{q}})]$.

It is useful to note that Eq. (8.37) is, in some sense, closely related to the relation in Eq. (2.34). However, Eq. (2.34) is true in general, while Eq. (8.37) was derived for a special case; thus it is not always true that the thermal effects can simply be obtained by replacing the zero-temperature time-ordered propagator by Eq. (8.37), even if surprisingly often such a simple recipy does indeed function.

Fermionic case

Let us next consider two-point correlation functions built out of fermionic operators[20]. In contrast to the bosonic case, however, we take for generality the density matrix to be of the form $\hat{\rho} = \mathcal{Z}^{-1}\exp[-\beta(\hat{H} - \mu\hat{Q})]$.

We denote the operators which appear in the two-point functions by $\hat{j}_\alpha(x)$, $\hat{\bar{j}}_\beta(x)$. They could be elementary field operators, in which case the indices α, β label Dirac and/or flavour components, but they could also be composite operators consisting of a product of elementary field operators. Nevertheless, we assume the validity of the relation

$$
[\hat{j}_\alpha(0, \mathbf{x}), \hat{Q}] \;\; = \;\; -\hat{j}_\alpha(0, \mathbf{x}) \; . \qquad (8.38)
$$

To motivate this, note that for $\hat{j}_\alpha \equiv \hat{\psi}_\alpha$, $\hat{\bar{j}}_\beta = \hat{\bar{\psi}}_\beta$, the canonical commutation relations of Eq. (4.32),

$$
\{\hat{\psi}_\alpha(x^0, \mathbf{x}), \hat{\psi}^\dagger_\beta(x^0, \mathbf{y})\} = \delta^{(d)}(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta} \; , \qquad (8.39)
$$

and the expression for the conserved charge in Eq. (7.32),

$$
\hat{Q} = -\int \mathrm{d}^d\mathbf{x}\, \hat{\bar{\psi}}\gamma_0\hat{\psi} = -\int \mathrm{d}^d\mathbf{x}\, \hat{\psi}^\dagger_\alpha\hat{\psi}_\alpha \; , \qquad (8.40)
$$

as well as the identity $[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} = \{\hat{A}, \hat{B}\}\hat{C} - \hat{B}\{\hat{A}, \hat{C}\}$, imply that in this case Eq. (8.38) is indeed satisfied. Eq. (8.38) implies that

$$
e^{\beta\mu\hat{Q}}\hat{j}_\alpha(0, \mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\beta\mu)^n(\hat{Q})^n\hat{j}_\alpha(0, \mathbf{x}) = \sum_{n=0}^{\infty}\frac{1}{n!}(\beta\mu)^n\hat{j}_\alpha(0, \mathbf{x})(\hat{Q} + 1)^n = \hat{j}_\alpha(0, \mathbf{x})e^{\beta\mu\hat{Q}}e^{\beta\mu} \; , \qquad (8.41)
$$

and consequently that

$$
\begin{aligned}
\Big\langle \hat{j}_\alpha(\beta, \mathbf{x})\hat{\bar{j}}_\beta(0, \mathbf{0})\Big\rangle \;\; &= \;\; \frac{1}{\mathcal{Z}}\mathrm{Tr}\Big[e^{-\beta(\hat{H} - \mu\hat{Q})}e^{\beta\hat{H}}\hat{j}_\alpha(0, \mathbf{x})e^{-\beta\hat{H}}\hat{\bar{j}}_\beta(0, \mathbf{0})\Big] \\
&= \;\; \frac{1}{\mathcal{Z}}\mathrm{Tr}\Big[\hat{j}_\alpha(0, \mathbf{x})e^{\beta\mu}e^{-\beta(\hat{H} - \mu\hat{Q})}\hat{\bar{j}}_\beta(0, \mathbf{0})\Big] \\
&= \;\; \frac{1}{\mathcal{Z}}e^{\mu\beta}\mathrm{Tr}\Big[\hat{j}_\alpha(0, \mathbf{x})e^{-\beta(\hat{H} - \mu\hat{Q})}\hat{\bar{j}}_\beta(0, \mathbf{0})\Big] \\
&= \;\; e^{\mu\beta}\Big\langle \hat{\bar{j}}_\beta(0, \mathbf{0})\hat{j}_\alpha(0, \mathbf{x})\Big\rangle \; . \qquad (8.42)
\end{aligned}
$$

This is a fermionic version of the KMS relation.

With this setting, we can again define various classes of correlation functions, like in the bosonic case. The "physical" correlators are now set up as

$$\Pi^{>}_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \hat{j}_\alpha(x) \hat{\bar{j}}_\beta(0) \Big\rangle \,, \tag{8.43}$$

$$\Pi^{<}_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle -\hat{\bar{j}}_\beta(0) \hat{j}_\alpha(x) \Big\rangle \,, \tag{8.44}$$

$$\rho_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \frac{1}{2} \Big\{ \hat{j}_\alpha(x), \hat{\bar{j}}_\beta(0) \Big\} \Big\rangle \,, \tag{8.45}$$

$$\Delta_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \frac{1}{2} \Big[\hat{j}_\alpha(x), \hat{\bar{j}}_\beta(0) \Big] \Big\rangle \,, \tag{8.46}$$

where $\rho_{\alpha\beta}$ is the spectral function, while retarded and advanced correlators can be defined as

$$\Pi^{R}_{\alpha\beta}(Q) \equiv i \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \Big\{ \hat{j}_\alpha(x), \hat{\bar{j}}_\beta(0) \Big\} \theta(t) \Big\rangle \,, \tag{8.47}$$

$$\Pi^{A}_{\alpha\beta}(Q) \equiv i \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle -\Big\{ \hat{j}_\alpha(x), \hat{\bar{j}}_\beta(0) \Big\} \theta(-t) \Big\rangle \,. \tag{8.48}$$

On the other hand, the time-ordered correlation function reads

$$\Pi^{T}_{\alpha\beta}(Q) \equiv \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \Big\langle \hat{j}_\alpha(x) \hat{\bar{j}}_\beta(0) \theta(t) - \hat{\bar{j}}_\beta(0) \hat{j}_\alpha(x) \theta(-t) \Big\rangle \,, \tag{8.49}$$

while the Euclidean correlator is

$$\Pi^{E}_{\alpha\beta}(\tilde{Q}) \equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x}\, e^{(i\tilde{q}_0-\mu)\tau - i\mathbf{q}\cdot\mathbf{x}} \Big\langle \hat{j}_\alpha(\tilde{x}) \hat{\bar{j}}_\beta(0) \Big\rangle \,. \tag{8.50}$$

Note again that the Euclidean correlator is time-ordered by definition, and can be computed with standard imaginary-time functional integrals in the Matsubara formalism. The Matsubara frequencies $\tilde{q}_0$ in Eq. (8.50) are fermionic, and the additional factor in the exponential in Eq. (8.50) is chosen in order to cancel the extra multiplicative factor in Eq. (8.42).

We can establish relations between the different Green's functions just like in the bosonic case:

$$\begin{aligned}
\Pi^{>}_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \mathrm{Tr} \Big[e^{-\beta\hat{H}+i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} e^{\beta\mu\hat{Q}} \hat{j}_\alpha(0,\mathbf{x}) e^{-i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|} \hat{\bar{j}}_\beta(0,\mathbf{0}) \Big] \\
&= \frac{1}{\mathcal{Z}} \sum_{m,n} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} e^{(-\beta+it)E_m} e^{-itE_n} e^{\beta\mu} \langle m|\hat{j}_\alpha(0,\mathbf{x}) e^{\beta\mu\hat{Q}}|n\rangle \, \langle n|\hat{\bar{j}}_\beta(0,\mathbf{0})|m\rangle \\
&= \frac{1}{\mathcal{Z}} \int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}} \sum_{m,n} e^{-\beta(E_m-\mu)}\, 2\pi\, \delta(q^0 + E_m - E_n) \langle m|\hat{j}_\alpha(0,\mathbf{x}) e^{\beta\mu\hat{Q}}|n\rangle \, \langle n|\hat{\bar{j}}_\beta(0,\mathbf{0})|m\rangle \,,
\end{aligned} \tag{8.51}$$

$$\begin{aligned}
\Pi^{<}_{\alpha\beta}(Q) &= -\frac{1}{\mathcal{Z}} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \mathrm{Tr} \Big[e^{-\beta\hat{H}} e^{\beta\mu\hat{Q}} \underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|} \hat{\bar{j}}_\beta(0,\mathbf{0}) e^{i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} \hat{j}_\alpha(0,\mathbf{x}) e^{-i\hat{H}t} \Big] \\
&= -\frac{1}{\mathcal{Z}} \sum_{m,n} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} e^{(-\beta-it)E_n} e^{itE_m} \langle n|\hat{\bar{j}}_\beta(0,\mathbf{0})|m\rangle \, \langle m|\hat{j}_\alpha(0,\mathbf{x}) e^{\beta\mu\hat{Q}}|n\rangle \\
&= -\frac{1}{\mathcal{Z}} \int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}} \sum_{m,n} e^{-\beta E_n}\, 2\pi\, \underbrace{\delta(q^0 + E_m - E_n)}_{E_n = E_m + q^0} \langle m|\hat{j}_\alpha(0,\mathbf{x}) e^{\beta\mu\hat{Q}}|n\rangle \, \langle n|\hat{\bar{j}}_\beta(0,\mathbf{0})|m\rangle \\
&= -e^{-\beta(q^0+\mu)}\, \Pi^{>}_{\alpha\beta}(Q) \,.
\end{aligned} \tag{8.52}$$

Consequently, $\rho_{\alpha\beta}(Q) = [\Pi^{>}_{\alpha\beta}(Q) - \Pi^{<}_{\alpha\beta}(Q)]/2$ and, conversely,

$$\Pi^{>}_{\alpha\beta}(Q) = 2[1 - n_{\mathrm{F}}(q^0+\mu)]\rho_{\alpha\beta}(Q) \,, \quad \Pi^{<}_{\alpha\beta}(Q) = -2n_{\mathrm{F}}(q^0+\mu)\rho_{\alpha\beta}(Q) \,, \tag{8.53}$$

where $n_{\rm F}(x) \equiv 1/[\exp(\beta x)+1]$. Moreover, $\Delta_{\alpha\beta}(Q) = [1-2n_{\rm F}(q^0+\mu)]\rho_{\alpha\beta}(Q)$, in which relation the combination $[1-2n_{\rm F}(q^0+\mu)]$ has a specific symmetry property: $[1-2n_{\rm F}(-q^0-\mu)] = -[1-2n_{\rm F}(q^0+\mu)]$.

The relation of Π^R, Π^A and Π^T to the spectral function can now be derived in complete analogy with Eqs. (8.18)–(8.23). For brevity we only cite the final results:

$$\Pi^R_{\alpha\beta}(Q) = \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{p^0-q^0-i0^+} \;, \quad \Pi^A_{\alpha\beta}(Q) = \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{p^0-q^0+i0^+} \;, \tag{8.54}$$

$$\begin{aligned} \Pi^T_{\alpha\beta}(Q) &= \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{i\rho_{\alpha\beta}(p^0,\mathbf{q})}{q^0-p^0+i0^+} - 2n_{\rm F}(q^0+\mu)\rho_{\alpha\beta}(q^0,\mathbf{q}) \\ &= -i\Pi^R_{\alpha\beta}(Q) + \Pi^<_{\alpha\beta}(Q) \;. \end{aligned} \tag{8.55}$$

Note that when written in a "generic form", where no distribution functions are visible, the end results are identical with the bosonic ones.

Finally, writing the argument inside the τ-integration in Eq. (8.50) as a Wick rotation of the inverse Fourier transform of the left-hand side of Eq. (8.43), inserting Eq. (8.53), and changing orders of integration, we get

$$\begin{aligned} \Pi^E_{\alpha\beta}(\tilde{Q}) &= \int_0^\beta {\rm d}\tau\, e^{(i\tilde{q}_0-\mu)\tau} \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{2\pi} e^{-p^0\tau} \Pi^>_{\alpha\beta}(p^0,\mathbf{q}) \\ &= \int_0^\beta {\rm d}\tau\, e^{(i\tilde{q}_0-\mu)\tau} \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{2\pi} e^{-p^0\tau} \frac{2e^{\beta(p^0+\mu)}}{e^{\beta(p^0+\mu)}+1} \rho_{\alpha\beta}(p^0,\mathbf{q}) \\ &= \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{e^{\beta(p^0+\mu)}}{e^{\beta(p^0+\mu)}+1} \rho_{\alpha\beta}(p^0,\mathbf{q}) \int_0^\beta {\rm d}\tau\, e^{(i\tilde{q}_0-\mu-p^0)\tau} \\ &= \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{e^{\beta(p^0+\mu)}}{e^{\beta(p^0+\mu)}+1} \rho_{\alpha\beta}(p^0,\mathbf{q}) \left[\frac{e^{(i\tilde{q}_0-\mu-p^0)\tau}}{i\tilde{q}_0-\mu-p^0} \right]_0^\beta \\ &= \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{e^{\beta(p^0+\mu)}}{e^{\beta(p^0+\mu)}+1} \rho_{\alpha\beta}(p^0,\mathbf{q}) \frac{-e^{-\beta(p^0+\mu)}-1}{i\tilde{q}_0-\mu-p^0} \\ &= \int_{-\infty}^{\infty} \frac{{\rm d}p^0}{\pi} \frac{\rho_{\alpha\beta}(p^0,\mathbf{q})}{p^0-i[\tilde{q}_0+i\mu]} \;. \end{aligned} \tag{8.56}$$

Like in the bosonic case, this relation can be inverted by making use of Eq. (8.21),

$$\rho(q^0,\mathbf{q}) = \frac{1}{2i} \,{\rm Disc}\, \Pi^E_{\alpha\beta}(\tilde{q}_0+i\mu \to -i[q^0+i0^+],\mathbf{q}) = \frac{1}{2i} \,{\rm Disc}\, \Pi^R_{\alpha\beta}(Q) \;, \tag{8.57}$$

where the discontinuity is defined like in Eq. (8.28).

We also note that the fermionic Matsubara sum over the structure in Eq. (8.56) can be carried out explicitly. This could be verified by making use of Eq. (4.76), in analogy with the bosonic analysis in Eqs. (8.31), (8.32), but let us for a change proceed in another way. We may recall, first of all, that

$$T\sum_{\omega_{\rm b}} e^{i\omega_{\rm b}\tau} = \delta(\tau \bmod \beta) \;. \tag{8.58}$$

According to Eq. (4.54), viz. $S_{\rm f}(T) = 2S_{\rm b}\Big(\frac{T}{2}\Big) - S_{\rm b}(T)$, we can write

$$T\sum_{\omega_{\rm f}} e^{i\omega_{\rm f}\tau} = 2\delta(\tau \bmod 2\beta) - \delta(\tau \bmod \beta) \;. \tag{8.59}$$

We assume for a moment that $p^0+\mu > 0$. Employing the representation

$$\frac{1}{\alpha+i\beta} = \int_0^\infty {\rm d}s\, e^{-(\alpha+i\beta)s} \;, \quad \alpha > 0 \;, \tag{8.60}$$

and inserting subsequently Eq. (8.59), we get

$$\begin{aligned}
T\sum_{\omega_{\rm f}}\frac{1}{p^0+\mu-i\omega_{\rm f}}e^{i\omega_{\rm f}\tau} &= \int_0^\infty {\rm d}s\, T\sum_{\omega_{\rm f}} e^{i\omega_{\rm f}\tau - p^0 s-\mu s+i\omega_{\rm f}s} \\
&= \int_0^\infty {\rm d}s\, e^{-(p^0+\mu)s}\Big[2\delta(\tau+s \bmod 2\beta)-\delta(\tau+s \bmod \beta)\Big] \\
&= 2\sum_{n=1}^\infty e^{-(p^0+\mu)(-\tau+2\beta n)} - \sum_{n=1}^\infty e^{-(p^0+\mu)(-\tau+\beta n)} \\
&= e^{(p^0+\mu)\tau}\Bigg[\underbrace{2\underbrace{\sum_{n=1}^\infty e^{-2\beta(p^0+\mu)n}}_{\frac{e^{-2\beta(p^0+\mu)}}{1-e^{-2\beta(p^0+\mu)}}} - \underbrace{\sum_{n=1}^\infty e^{-\beta(p^0+\mu)n}}_{\frac{e^{-\beta(p^0+\mu)}}{1-e^{-\beta(p^0+\mu)}}}}_{\frac{2}{(e^{\beta(p^0+\mu)}-1)(e^{\beta(p^0+\mu)}+1)}-\frac{1}{e^{\beta(p^0+\mu)}-1}}\Bigg] \\
&= -e^{(p^0+\mu)\tau}n_{\rm F}(p^0+\mu)\;, \qquad (8.61)
\end{aligned}$$

where we assumed $0<\tau<\beta$. As an immediate consequence,

$$T\sum_{\omega_{\rm f}}\frac{1}{p^0+\mu-i\omega_{\rm f}}e^{-i\omega_{\rm f}\tau} = -T\sum_{\omega_{\rm f}}\frac{1}{p^0+\mu-i\omega_{\rm f}}e^{i\omega_{\rm f}(\beta-\tau)} = e^{(\beta-\tau)(p^0+\mu)}n_{\rm F}(p^0+\mu)\;. \qquad (8.62)$$

Furthermore, it is not too difficult to show (by substituting $\omega_{\rm f}\to-\omega_{\rm f}$) that these relations continue to hold also for $p^0+\mu<0$.

As a consequence of Eq. (8.61), let us note that

$$\begin{aligned}
T\sum_{\omega_{\rm f}}\frac{e^{i(\omega_{\rm f}-i\mu)\tau}}{(\omega_{\rm f}-i\mu)^2+\omega^2} &= e^{\mu\tau}T\sum_{\omega_{\rm f}}e^{i\omega_{\rm f}\tau}\frac{1}{(\omega-i\omega_{\rm f}-\mu)(\omega+i\omega_{\rm f}+\mu)} \\
&= e^{\mu\tau}T\sum_{\omega_{\rm f}}e^{i\omega_{\rm f}\tau}\frac{1}{2\omega}\left[\frac{1}{\omega+\mu+i\omega_{\rm f}}+\frac{1}{\omega-\mu-i\omega_{\rm f}}\right] \\
&= \frac{e^{\mu\tau}}{2\omega}\Big[e^{-(\omega+\mu)\tau}n_{\rm F}(-\omega-\mu)-e^{(\omega-\mu)\tau}n_{\rm F}(\omega-\mu)\Big] \\
&= \frac{e^{\mu\tau}}{2\omega}\Big[e^{(\beta-\tau)(\omega+\mu)}n_{\rm F}(\omega+\mu)-e^{\tau(\omega-\mu)}n_{\rm F}(\omega-\mu)\Big] \\
&= \frac{1}{2\omega}\Big[n_{\rm F}(\omega+\mu)e^{(\beta-\tau)\omega+\beta\mu}-n_{\rm F}(\omega-\mu)e^{\tau\omega}\Big]\;, \qquad (8.63)
\end{aligned}$$

which constitutes a generalization of Eq. (4.76) to the case of a finite chemical potential.

Example

Let us illustrate some of the relations obtained by considering the structure of the free fermion propagator in the presence of a chemical potential. With fermions, in the presence of $\mu\neq0$, one unfortunately has to be extremely careful with definitions. Suppressing spatial momenta and indices, Eq. (5.47) and the presence of a chemical potential *à la* Eq. (7.34) imply that the free propagator can be written in the form

$$\langle\hat\psi(\tau)\hat{\bar\psi}(0)\rangle = T\sum_{\tilde p_{0{\rm f}}}e^{i(\tilde p_0-i\mu)\tau}\frac{-iA(\tilde p_0-i\mu)+B}{(\tilde p_0-i\mu)^2+E^2}\;, \qquad (8.64)$$

where an additional exponential has been inserted into the Fourier transform, in order to respect the property in Eq. (8.42). The correlator in Eq. (8.50) then becomes

$$\begin{aligned}\Pi^E(\tilde{q}_0) &= \int_0^\beta \mathrm{d}\tau\, e^{(i\tilde{q}_0-\mu)\tau} T\sum_{\tilde{p}_{0\mathrm{f}}} e^{i(\tilde{p}_0-i\mu)\tau}\frac{-iA(\tilde{p}_0-i\mu)+B}{(\tilde{p}_0-i\mu)^2+E^2} \\ &= \frac{iA(\tilde{q}_0+i\mu)+B}{(\tilde{q}_0+i\mu)^2+E^2}\,. \end{aligned} \tag{8.65}$$

According to Eq. (8.57),

$$\begin{aligned}\rho(q^0) &= \frac{1}{2i}\left[\frac{A(q^0+i0^+)+B}{-(q^0+i0^+)^2+E^2}-\frac{A(q^0-i0^+)+B}{-(q^0-i0^+)^2+E^2}\right] \\ &= \frac{1}{2i}\frac{1}{2E}\left[\frac{A(q^0+i0^+)+B}{q^0+i0^++E}-\frac{A(q^0+i0^+)+B}{q^0+i0^+-E}-\frac{A(q^0-i0^+)+B}{q^0-i0^++E}+\frac{A(q^0-i0^+)+B}{q^0-i0^+-E}\right] \\ &= \frac{1}{2i}\frac{1}{2E}\left[\frac{-AE+B}{q^0+i0^++E}-\frac{AE+B}{q^0+i0^+-E}-\frac{-AE+B}{q^0-i0^++E}+\frac{AE+B}{q^0-i0^+-E}\right] \\ &= \frac{\pi}{4E}\Big[\delta(q^0-E)\Big(AE+B+AE+B\Big)+\delta(q^0+E)\Big(AE-B+AE-B\Big)\Big] \\ &= \frac{\pi}{2E}\Big[\delta(q^0-E)-\delta(q^0+E)\Big](Aq^0+B)\,. \end{aligned} \tag{8.66}$$

Note that the tree-level spectral function is independent of the temperature and of the chemical potential! The retarded propagator reads

$$\Pi^R(q^0)=\frac{A(q^0+i0^+)+B}{-(q^0+i0^+)^2+E^2}\,, \tag{8.67}$$

and, from Eqs. (8.53), (8.55), the time-ordered propagator can be determined after a few steps:

$$\begin{aligned}\Pi^T(q^0) &= (Aq^0+B)\Big\{\frac{-i}{2E}\Big(\frac{1}{E-q^0-i0^+}+\frac{1}{E+q^0+i0^+}\Big) \\ &\quad -\frac{2\pi}{2E}n_{\mathrm{F}}(q^0+\mu)\Big[\delta(q^0-E)-\delta(q^0+E)\Big]\Big\} \\ &= \frac{Aq^0+B}{2E}\Big\{-iP\Big(\frac{1}{E-q^0}\Big)-iP\Big(\frac{1}{E+q^0}\Big) \\ &\quad +\pi\delta(q^0-E)\Big[1-2n_{\mathrm{F}}(q^0+\mu)\Big]-\pi\delta(q^0+E)\Big[1-2n_{\mathrm{F}}(q^0+\mu)\Big]\Big\} \\ &= \frac{Aq^0+B}{2E}\Big\{-iP\Big(\frac{2E}{E^2-(q^0)^2}\Big) \\ &\quad +\pi\delta(q^0-E)\Big[1-2n_{\mathrm{F}}(q^0+\mu)\Big]+\pi\delta(q^0+E)\Big[1-2n_{\mathrm{F}}(-q^0-\mu)\Big]\Big\} \\ &= \frac{Aq^0+B}{2E}\Big\{-iP\Big(\frac{2E}{E^2-(q^0)^2}\Big)+2E\pi\delta\Big((q^0)^2-E^2\Big) \\ &\quad -2\pi\Big[\delta(q^0-E)n_{\mathrm{F}}(q^0+\mu)]+\delta(q^0+E)n_{\mathrm{F}}(-q^0-\mu)\Big]\Big\} \\ &= (Aq^0+B)\Big\{\frac{i}{(q^0)^2-E^2+i0^+}-2\pi\,\delta\Big((q^0)^2-E^2\Big)n_{\mathrm{F}}\Big(|q^0|+\mathrm{sign}(q^0)\mu\Big)\Big\}\,. \end{aligned} \tag{8.68}$$

All temperature and density effects are again seen to reside in an on-shell part and, to some extent, one could hope to account for finite temperature effects simply by replacing free zero-temperature propagators through Eq. (8.68); the proper procedure, however, is to carry out the analytic continuation for the *complete observable* considered, and this may not always amount to the simple replacement through Eq. (8.68) of all the free propagators appearing in the graph.