7.3. Dirac fermion with a finite chemical potential

The Lagrangian of a Dirac fermion,

$$\mathcal{L}_M = \bar{\psi}_A(i\slashed{D}_{AB} - m\,\delta_{AB})\psi_B\,, \quad \slashed{D}_{AB} = \gamma^\mu(\delta_{AB}\partial_\mu - igA_\mu^a T_{AB}^a)\,, \tag{7.29}$$

possesses a *global symmetry*,

$$\psi_A \to e^{i\alpha}\psi_A\,, \quad \bar{\psi}_A \to e^{-i\alpha}\bar{\psi}_A\,, \tag{7.30}$$

in addition to the usual non-Abelian (local) gauge symmetry. Therefore there is an associated conserved quantity, and we can again consider the behaviour of the system in the presence of a chemical potential.

The conserved Noether current reads

$$\begin{aligned} J_\mu &= \frac{\partial\mathcal{L}_M}{\partial(\partial^\mu\psi_A)}\frac{\delta\psi_A}{\delta\alpha} \\ &= \bar{\psi}_A\,i\gamma_\mu\,i\psi_A = -\bar{\psi}_A\gamma_\mu\psi_A\,. \end{aligned} \tag{7.31}$$

The corresponding charge is $Q = \int_{\mathbf{x}} J_0$, and as an operator it commutes with the Hamiltonian, $[\hat{H}, \hat{Q}] = 0$. Therefore, like with the scalar field theory, we can treat the combination $\hat{H} - \mu\hat{Q}$ as an "effective" Hamiltonian, and directly write down the corresponding path integral, by simply adding

$$-\mu\,Q = \mu\int \mathrm{d}^d\mathbf{x}\,\bar{\psi}_A\gamma_0\psi_A \tag{7.32}$$

to the Euclidean action. The path integral thereby reads

$$\mathcal{Z}(T, \mu) = \int_{\text{antiperiodic}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x}\,\bar{\psi}[\tilde{\gamma}_\mu\tilde{D}_\mu + \tilde{\gamma}_0\,\mu + m]\psi\right\}\,. \tag{7.33}$$

For perturbation theory, let us consider the quadratic part of the Euclidean action. Going to momentum space, we get

$$S_E = \sum_{\tilde{P}_{\text{f}}} \tilde{\bar{\psi}}(\tilde{P})[i\tilde{\gamma}_0\omega_n + i\tilde{\gamma}_i p_i + \tilde{\gamma}_0\mu + m]\tilde{\psi}(\tilde{P})\,. \tag{7.34}$$

Therefore, just like in the previous section, the existence of a chemical potential corresponds simply to the shift $\omega_n \to \omega_n - i\mu$ of the Matsubara frequencies. Everything else — propagators, Feynman rules, etc — remains unchanged.

In particular, let us write down the free energy density of a single free Dirac fermion. Compared with a complex scalar field, there is an overall factor -2 (rather than -4 in Eq. (4.51), where we compared with a real scalar field). Otherwise, the chemical potential appears in identical ways in Eqs. (7.16) and Eq. (7.34), so Eq. (4.54), $S_{\text{f}}(T) = 2S_{\text{b}}\left(\frac{T}{2}\right) - S_{\text{b}}(T)$, continues to apply. Employing it with Eq. (7.18) we get

$$\begin{aligned} f(T, \mu) &= -2\int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}\left\{E + T\left[\ln\left(1 - e^{-2\beta(E+\mu)}\right) + \ln\left(1 - e^{-2\beta(E-\mu)}\right)\right.\right. \\ &\qquad\qquad \left.\left. -\ln\left(1 - e^{-\beta(E+\mu)}\right) - \ln\left(1 - e^{-\beta(E-\mu)}\right)\right]\right\}\bigg|_{E=\sqrt{\mathbf{p}^2+m^2}} \\ &= -2\int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}\left\{E + T\left[\ln\left(1 + e^{-\frac{E+\mu}{T}}\right) + \ln\left(1 + e^{-\frac{E-\mu}{T}}\right)\right]\right\}\bigg|_{E=\sqrt{\mathbf{p}^2+m^2}}\,. \tag{7.35} \end{aligned}$$

This integral is well-defined for any μ; thus free elementary fermions do not suffer from infrared problems even with $\mu \neq 0$, and do not undergo any type of condensation.

7.4. How about chemical potentials for gauge symmetries?

We already saw, after Eq. (7.2), that a chemical potential has some relation to a gauge field $\tilde{A}_0$. However, in cases like QCD, a chemical potential has no colour structure (i.e. it is an identity matrix in colour space), while $\tilde{A}_0$ is a traceless matrix in colour space. On the other hand, in QED, $\tilde{A}_0$ is not traceless. In fact, in QED, the gauge symmetry is nothing but a local version of that in Eq. (7.30). We may therefore ask whether we can associate a chemical potential with the electric charge of QED, and what is the precise relation of $\tilde{A}_0$ and μ in this case.

Let us first recall what happens in such a situation physically. A non-zero chemical potential corresponds to the a system which is *charged*. Moreover, if we want to describe it perturbatively with the QED Lagrangian, we had better choose a system where the charge carriers (particles) are essentially free; such a system could be a metal or a plasma. In this situation, though, the free charge carriers interact repulsively with a long-range force, and hence all the net charge resides *on the surface*. In other words, the homogeneous "bulk" of the medium is *neutral* (i.e. has no free charge). On the other hand, the charged body as a whole has a non-zero electric potential, V_0, with respect to the neutral ground.

Let us now try to understand how to reproduce this behaviour directly from the partition function, Eq. (7.33), adapted to QED:

$$\mathcal{Z}(T,\mu) = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi \, \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x} \left[\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}\left(\tilde{\gamma}_0(\partial_\tau - ie\tilde{A}_0 + \mu) + \gamma_i D_i + m \right)\psi \right] \right\}. \tag{7.36}$$

The usual boundary conditions over the time direction are assumed, even if not shown explicitly. The basic claim is, according to the physical picture above, that if we by construction assume the system to be *homogeneous*, i.e. consider the "bulk" situation, then the partition function *should not depend on μ*, in order to ensure the neutrality that we expect:

$$\rho \propto \frac{\partial f}{\partial \mu} = 0 \,. \tag{7.37}$$

How does this arise?

The key observation is that we should again think of the system in terms of an effective potential, like in Eq. (7.20). The role of the condensate is now given to the field $\tilde{A}_0$; let us denote it by $\bar{\tilde{A}}_0$. The last integral to be carried out is

$$\mathcal{Z}(T,\mu) = \int_{-\infty}^{\infty} \mathrm{d}\bar{\tilde{A}}_0 \, \exp\left\{ -\frac{V}{T} V_{\mathrm{eff}}(\bar{\tilde{A}}_0) \right\} . \tag{7.38}$$

Now, we can deduce from Eq. (7.36) that μ can only appear in the combination $-ie\bar{\tilde{A}}_0 + \mu$, so that $V_{\mathrm{eff}}(\bar{\tilde{A}}_0) = f(\bar{\tilde{A}}_0 + i\mu/e)$. Moreover, we know from Eq. (6.36) that in a large volume and large temperature,

$$V_{\mathrm{eff}}(\bar{\tilde{A}}_0) \approx \frac{1}{2} m_E^2 (\bar{\tilde{A}}_0 + i\mu/e)^2 + \mathcal{O}(\bar{\tilde{A}}_0 + i\mu/e)^4 \,, \tag{7.39}$$

where $m_E^2 \sim e^2 T^2$. (The exact effective potential, which indeed only has a quadratic and a quartic term, can be deduced from the results of Exercise 10.) In the infinite-volume limit, the integral in Eq. (7.38) can be carried out by making use of the *saddle point approximation*. The saddle point is located in the complex plane at the position where $V'_{\mathrm{eff}}(\bar{\tilde{A}}_0) = 0$, i.e. at $\bar{\tilde{A}}_0 = -i\mu/e$. The value of the potential at the saddle point, as well as the Gaussian integral around it, are clearly independent of μ. This, then, leads to Eq. (7.37).

It is interesting to note that the saddle point is purely imaginary. Recalling the relation of Minkowskian and Euclidean A_0 from page 59, this corresponds to a real Minkowskian A_0. Thus there indeed is a real electric potential $V_0 \propto \mu$, just as we expected physically.

7.5. Exercise 10

The free energy density of a single Dirac fermion,

$$f(T,\mu) = -2\!\!\!\sum_{\tilde{P}_{\mathrm{f}}}\!\!\!\!\!\!\!\!\int \ln[(\omega_n - i\mu)^2 + E^2]\,, \tag{7.40}$$

can be computed explicitly for the case $m = 0$ (i.e. $E = |\mathbf{p}|$). Show that, subtracting the vacuum part, the result is

$$f(T,\mu) = -\left(\frac{7\pi^2 T^4}{180} + \frac{\mu^2 T^2}{6} + \frac{\mu^4}{12\pi^2}\right). \tag{7.41}$$

Solution to Exercise 10

We start from the expression in Eq. (7.35), subtracting the T-independent vacuum term and setting $m = 0, d = 3$:

$$\begin{aligned}
f(T,\mu) &= -2T\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left[\ln\!\left(1 + e^{-\frac{|\mathbf{p}|+\mu}{T}}\right) + \ln\!\left(1 + e^{-\frac{|\mathbf{p}|-\mu}{T}}\right)\right]\\
&= -\frac{T^4}{\pi^2}\int_0^\infty \mathrm{d}x\, x^2\left[\ln\!\left(1 + e^{-x-y}\right) + \ln\!\left(1 + e^{-x+y}\right)\right],
\end{aligned} \tag{7.42}$$

where we set $x \equiv |\mathbf{p}|/T$, $y \equiv \mu/T$ on the latter row, and carried out the angular integral.

A possible trick now is to expand the logarithms in Taylor series,

$$\ln(1 + z) = \sum_{n=1}^\infty (-1)^{n+1}\frac{z^n}{n}\,,\quad |z| < 1\,. \tag{7.43}$$

Assuming $y > 0$, this is indeed possible with the first term of Eq. (7.42), while in the second term a direct application is not possible. However, if $e^{-x+y} > 1$, we can write $1 + e^{-x+y} = e^{-x+y}(1 + e^{x-y})$, where $e^{x-y} < 1$; thereby the Taylor expansion can be written as

$$\ln\!\left(1 + e^{-x+y}\right) = \theta(x - y)\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}e^{-xn}e^{yn} + \theta(y - x)\left[y - x + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}e^{xn}e^{-yn}\right]. \tag{7.44}$$

Inserting this into Eq. (7.42), we get

$$\begin{aligned}
f(T,\mu) &= -\frac{T^4}{\pi^2}\Bigg\{\int_0^y \mathrm{d}x\left[yx^2 - x^3 + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}x^2\left(e^{xn}e^{-yn} + e^{-xn}e^{-yn}\right)\right]\\
&\quad + \int_y^\infty \mathrm{d}x\left[\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}x^2\left(e^{-xn}e^{yn} + e^{-xn}e^{-yn}\right)\right]\\
&= -\frac{T^4}{\pi^2}\Bigg\{\int_0^y \mathrm{d}x\left[yx^2 - x^3 + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}x^2\left(e^{xn}e^{-yn} - e^{-xn}e^{yn}\right)\right]\\
&\quad + \int_0^\infty \mathrm{d}x\left[\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}x^2\left(e^{-xn}e^{yn} + e^{-xn}e^{-yn}\right)\right]. \tag{7.45}
\end{aligned}$$

All the x-integrals can be carried out:

$$\int_0^y \mathrm{d}x\,(yx^2 - x^3) = \left(\frac{1}{3} - \frac{1}{4}\right)y^4 = \frac{1}{12}y^4\,, \tag{7.46}$$

$$\int_0^y \mathrm{d}x\, x^2 e^{\alpha x} = -\frac{2}{\alpha^3} + e^{\alpha y}\left(\frac{2}{\alpha^3} - \frac{2y}{\alpha^2} + \frac{y^2}{\alpha}\right), \tag{7.47}$$

$$\int_0^\infty \mathrm{d}x\, x^2 e^{-xn} = \frac{2}{n^3}\,. \tag{7.48}$$

Inserting these into Eq. (7.45) we get

$$
\begin{aligned}
f(T, \mu) &= -\frac{T^4}{\pi^2} \left\{ \frac{y^4}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[e^{-yn} \left(-\frac{2}{n^3} + e^{yn} \left(\frac{2}{n^3} - \frac{2y}{n^2} + \frac{y^2}{n} \right) \right) \right. \right. \\
&\quad \left. \left. - e^{yn} \left(\frac{2}{n^3} + e^{yn} \left(-\frac{2}{n^3} - \frac{2y}{n^2} - \frac{y^2}{n} \right) \right) + e^{yn} \frac{2}{n^3} + e^{-yn} \frac{2}{n^3} \right] \right\} \\
&= -\frac{T^4}{\pi^2} \left\{ \frac{y^4}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{4}{n^3} + \frac{2y^2}{n} \right] \right\},
\end{aligned}
\tag{7.49}
$$

where a remarkable cancellation took place. The sums can be carried out:

$$
\begin{aligned}
\eta(2) &\equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \\
&= \zeta(2) - \frac{2}{2^2} \zeta(2) = \frac{1}{2} \zeta(2) = \frac{\pi^2}{12},
\end{aligned}
\tag{7.50}
$$

$$
\begin{aligned}
\eta(4) &\equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots \\
&= \zeta(4) - \frac{2}{2^4} \zeta(4) = \frac{7}{8} \zeta(4) = \frac{7}{8} \frac{\pi^4}{90}.
\end{aligned}
\tag{7.51}
$$

Inserting into Eq. (7.49), we end up with

$$
f(T, \mu) = -\frac{T^4}{\pi^2} \left\{ \frac{y^4}{12} + \frac{\pi^2}{6} y^2 + \frac{7\pi^4}{180} \right\},
\tag{7.52}
$$

which after the substitution $y = \mu/T$ reproduces Eq. (7.41).