

7. Finite density

Let us consider a system which possesses some conserved charge, Q . We assume the conserved charge to be *additive*, i.e. the charge can in principle have any (integer) value. Physical examples of possible Q 's include:

- the baryon number B and the lepton number L . (In fact, within the Standard Model, the combination $B + L$ is not conserved because of an anomaly¹⁹, so that strictly speaking only the linear combination $B - L$ is conserved; however, in practice the rate of violation of $B + L$ is exponentially small, so that we can treat both B and L as conserved quantities.)
- if weak interactions are switched off (i.e., if we inspect phenomena at a time scale well below 10^{-10} s, or distances well below 1 cm within the collision region of a particle experiment), then quantities like the strangeness S and the isospin I are also conserved.
- in some supersymmetric theories, there is a quantity called the R -charge which is conserved.
- in non-relativistic field theories, the particle number N is conserved.

The case of a conserved Q turns out to be analogous to the case of gauge fields, treated on p. 58; indeed, as we will see, the introduction of a chemical potential, μ , as a conjugate variable to Q , is closely related to the introduction of the gauge fields, \tilde{A}_0 , that were needed for imposing the Gauss law, “ $Q = 0$ ”, in the gauge field case. However, in contrast to that situation, we will work in a grand canonical ensemble in the following, so that the quantum mechanical partition function is in general of the type

$$\mathcal{Z}(T, \mu) \equiv \text{Tr} \left[e^{-\beta(\hat{H} - \mu\hat{Q})} \right]. \quad (7.1)$$

On p. 58 the projection operator $\delta_{\hat{Q}, \hat{0}}$ was effectively imposed as

$$\delta_{\hat{Q}, \hat{0}} = \int_{-\pi}^{\pi} \frac{dc}{2\pi} e^{ic\hat{Q}}, \quad (7.2)$$

with $c \propto \tilde{A}_0$. Comparing Eqs. (7.1), (7.2), we see that a chemical potential corresponds to something like a *constant purely imaginary gauge field* \tilde{A}_0 .

Now, let us go back to classical field theory for a moment, and recall that if the system possesses a *global* U(1) symmetry, then there exists, according to Noether’s theorem, a conserved global current, J_μ . The integral of the zero-component of the current, i.e. charge density, over the spatial volume, defines the conserved charge,

$$Q \equiv \int d^d \mathbf{x} J_0(t, \mathbf{x}). \quad (7.3)$$

Conversely, we can assume that in order to describe a system which does have a conserved global charge, then there should exist a global U(1) symmetry in its field-theoretic description. In general this indeed is the case, and we will restrict to these situations in the following. (One notable exception is free field theory where, due to lack of interactions, particle number is conserved even without a global symmetry; another is that a discrete symmetry, $\phi \rightarrow -\phi$, may also lead to the concept of a generalized “parity”, which acts as a *multiplicative* quantum number, with possible values ± 1 ; however, in this case no non-trivial *charge density* $\rho = \langle \hat{Q} \rangle / V$ can be defined in the thermodynamic limit).

An immediate consequence of the “inverse” use of the Noether theorem is that a *real* scalar field cannot describe a system with an additive conserved charge. As the simplest example, let us therefore consider a system of a *complex scalar field*.

¹⁹G. ’t Hooft, *Symmetry breaking through Bell-Jackiw anomalies*, Phys. Rev. Lett. 37 (1976) 8.

7.1. Complex scalar field with a finite chemical potential

The classical Lagrangian of a complex scalar field reads

$$\mathcal{L}_M = \partial^\mu \phi^* \partial_\mu \phi - V(\phi), \quad (7.4)$$

where the potential has the form

$$V(\phi) \equiv m^2 \phi^* \phi + \lambda (\phi^* \phi)^2. \quad (7.5)$$

The system is symmetric in the global (position-independent) phase transformation

$$\phi \rightarrow e^{i\alpha} \phi, \quad \phi^* \rightarrow e^{-i\alpha} \phi^*, \quad (7.6)$$

where $\alpha \in \mathbb{R}$. The corresponding Noether current is defined as

$$\begin{aligned} J_\mu &\equiv \frac{\partial \mathcal{L}_M}{\partial(\partial^\mu \phi)} \frac{\delta \phi}{\delta \alpha} + \frac{\partial \mathcal{L}_M}{\partial(\partial^\mu \phi^*)} \frac{\delta \phi^*}{\delta \alpha} \\ &= \partial_\mu \phi^* i\phi - \partial_\mu \phi i\phi^* \\ &= i[\partial_\mu \phi^* \phi - \phi^* \partial_\mu \phi] = 2 \operatorname{Im}[\phi^* \partial_\mu \phi]. \end{aligned} \quad (7.7)$$

Let us note that the overall sign (i.e., what we call particles and what antiparticles) is a matter of convention; we could equally well have defined the global symmetry through $\phi \rightarrow e^{-i\alpha} \phi$, $\phi^* \rightarrow e^{i\alpha} \phi^*$, and then J_μ would have the opposite sign.

The first task now, as always, is to write down a path integral expression for the partition function in Eq. (7.1). Subsequently, we may try to evaluate the partition function, in order to see what kind of phenomena take place in this system.

In order to write down the path integral, we start from the known expression of \mathcal{Z} of a real scalar field *without chemical potential*, i.e. the generalization to field theory of Eq. (1.37):

$$\mathcal{Z} \propto \int_{\text{periodic}} \mathcal{D}\phi_1 \int \mathcal{D}\pi_1 \exp \left\{ - \int_0^\beta d\tau \int_{\mathbf{x}} \left[\frac{1}{2} \pi_1^2 - i\pi_1 \partial_\tau \phi_1 + \frac{1}{2} (\partial_i \phi_1)^2 + V(\phi_1) \right] \right\}, \quad (7.8)$$

where $\pi_1 = \partial \phi_1 / \partial t$. Here the combination $\frac{1}{2} \pi_1^2 + \frac{1}{2} (\partial_i \phi_1)^2 + V(\phi_1)$ is nothing but the classical Hamiltonian density, $\mathcal{H}(\pi_1, \phi_1)$.

In order to be able to make use of Eq. (7.8), let us rewrite the complex scalar field ϕ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $\phi_i \in \mathbb{R}$. Then

$$\partial^\mu \phi^* \partial_\mu \phi = \frac{1}{2} \partial^\mu \phi_1 \partial_\mu \phi_1 + \frac{1}{2} \partial^\mu \phi_2 \partial_\mu \phi_2, \quad \phi^* \phi = \frac{1}{2} (\phi_1^2 + \phi_2^2), \quad (7.9)$$

and the classical Hamiltonian density reads

$$\mathcal{H} = \frac{1}{2} \left[\pi_1^2 + \pi_2^2 + (\partial_i \phi_1)^2 + (\partial_i \phi_2)^2 + m^2 \phi_1^2 + m^2 \phi_2^2 \right] + \frac{1}{4} \lambda (\phi_1^2 + \phi_2^2)^2. \quad (7.10)$$

In order to go to the grand-canonical ensemble, we need to add from Eqs. (7.3), (7.7) the classical version of $-\mu \hat{Q}$ to the Hamiltonian, cf. Eq. (7.1):

$$\begin{aligned} -\mu Q &= -\mu \int_{\mathbf{x}} \operatorname{Im} \left[(\phi_1 - i\phi_2) (\partial_t \phi_1 + i\partial_t \phi_2) \right] \\ &= -\mu \int_{\mathbf{x}} (\phi_1 \pi_2 - \phi_2 \pi_1) = \int_{\mathbf{x}} \mu (\pi_1 \phi_2 - \pi_2 \phi_1). \end{aligned} \quad (7.11)$$

Since the charge can be expressed in terms of the canonical variables, nothing changes in the derivation of the path integral, and we can simply replace the Hamiltonian of Eq. (7.8) by the sum of Eqs. (7.10), (7.11).

Finally, we again carry out the Gaussing integrals over π_1, π_2 :

$$\int d\pi_1 \exp\left\{-\left[\frac{1}{2}\pi_1^2 + \pi_1\left(-i\frac{\partial\phi_1}{\partial\tau} + \mu\phi_2\right)\right]\right\} = \text{const.} \times \exp\left\{-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} + i\mu\phi_2\right)^2\right\}, \quad (7.12)$$

$$\int d\pi_2 \exp\left\{-\left[\frac{1}{2}\pi_2^2 + \pi_2\left(-i\frac{\partial\phi_2}{\partial\tau} - \mu\phi_1\right)\right]\right\} = \text{const.} \times \exp\left\{-\frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2\right\}. \quad (7.13)$$

Afterwards, we can go back to the complex notation:

$$\begin{aligned} & \frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} + i\mu\phi_2\right)^2 + \frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2 \\ &= \frac{1}{2}\left[\left(\frac{\partial\phi_1}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_2}{\partial\tau}\right)^2\right] + \underbrace{\mu \times i\left[\phi_2\frac{\partial\phi_1}{\partial\tau} - \phi_1\frac{\partial\phi_2}{\partial\tau}\right]}_{\phi\partial_\tau\phi^* - \phi^*\partial_\tau\phi} - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) \\ &= [(\partial_\tau - \mu)\phi^*][(\partial_\tau + \mu)\phi]. \end{aligned} \quad (7.14)$$

In total, then, the path integral representation for the grand canonical partition function reads

$$\mathcal{Z}(T, \mu) = C \int_{\text{periodic}} \mathcal{D}\phi \exp\left\{-\int_0^\beta d\tau \int_{\mathbf{x}} \left[(\partial_\tau - \mu)\phi^*(\partial_\tau + \mu)\phi + \partial_i\phi^*\partial_i\phi + m^2\phi^*\phi + \lambda(\phi^*\phi)^2\right]\right\}. \quad (7.15)$$

We observe that, as anticipated, μ appears in a way reminiscent of an imaginary gauge field \tilde{A}_0 .

Let us finally work out the properties of the free theory in the presence of μ . Going to momentum space, the quadratic part of the Euclidean action becomes

$$\begin{aligned} S_E &= \int_{\tilde{P}_b} \tilde{\phi}^*(\tilde{P}) \left[(-i\omega_n - \mu)(i\omega_n + \mu) + \mathbf{p}^2 + m^2\right] \tilde{\phi}(\tilde{P}) \\ &= \int_{\tilde{P}_b} \tilde{\phi}^*(\tilde{P}) \tilde{\phi}(\tilde{P}) \left[(\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2\right]. \end{aligned} \quad (7.16)$$

We observe that the chemical potential corresponds simply to a *shift of the Matsubara frequencies by a constant imaginary term* (this is the reason for considering a corresponding sum in Eq. (2.36)). In particular, the propagator reads

$$\langle \tilde{\phi}(\tilde{P}) \tilde{\phi}^*(\tilde{Q}) \rangle_0 = \delta(\tilde{P} - \tilde{Q}) \frac{1}{(\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2}, \quad (7.17)$$

while the free energy density is obtained from Eqs. (2.44), (2.49), after replacing $c \rightarrow -i\mu$ and noting that for a *complex* scalar field, all Fourier modes are independent, whereby Eqs. (2.44), (2.49) are to be multiplied by a factor 2:

$$f(T, \mu) = \int \frac{d^d\mathbf{p}}{(2\pi)^d} \left\{ E + T \left[\ln\left(1 - e^{-\frac{E+\mu}{T}}\right) + \ln\left(1 - e^{-\frac{E-\mu}{T}}\right) \right] \right\} \Big|_{E=\sqrt{\mathbf{p}^2+m^2}}. \quad (7.18)$$

We may wonder how the existence of $\mu \neq 0$ affects the infrared problem of finite-temperature field theory. In Sec. 2.6 we found that the high-temperature expansion ($T \gg m$) of Eq. (7.18) has a peculiar structure, because Eq. (7.18) has a branch cut starting at $m^2 = 0$. From the last term in Eq. (7.18), we note that this problem has become *worse* in the presence of $\mu > 0$: now the integral is not defined even at a finite m if $\mu > m$, because then $\exp(-(E - \mu)/T) > 1$ at small $|\mathbf{p}|$. Of course, in an interacting theory, thermal corrections generate an effective mass $m_{\text{eff}}^2 \sim \lambda T^2$, which postpones the problem to larger μ . Nevertheless, at large enough μ it still exists.

It turns out that there is a significant *physics consequence* from this infrared problem: the existence of *Bose-Einstein condensation*, to which we now turn.

7.2. Effective potential and Bose-Einstein condensation

In order to treat properly complex scalar field theory with a chemical potential, two things need to be realized:

- (i) In contrast to gauge field theory, the infrared problem exists now even in the *non-interacting limit*. Therefore it cannot be cured by a non-perturbatively generated confinement scale. Rather, it corresponds to a strong dependence of the properties of the system on the volume, so we should indeed keep the volume finite.
- (ii) The chemical potential μ is a most useful quantity in theoretical computations, but it is somewhat “abstract” from a practical point of view; the physical properties of the system are typically best characterised not by μ but by the variable conjugate to μ , i.e. the number density of the conserved charge. Therefore, rather than trying to give μ some specific value, we should in the first place fix the number density.

So, let us put the system in a periodic box, $V = L_1 L_2 L_3$. The spatial momenta get discretized like in Eq. (2.9),

$$\mathbf{p} = 2\pi \left(\frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_3} \right), \quad (7.19)$$

with $n_i \in \mathbb{Z}$. The mode with $\omega_n^b = 0$, $\mathbf{p} = \mathbf{0}$ will be called *the condensate*, and denoted by $\bar{\phi}$.

We now rewrite the partition function in Eq. (7.15) as

$$\begin{aligned} \mathcal{Z}(T, \mu) &= \int_{-\infty}^{\infty} d\bar{\phi} \left\{ \int_{\text{periodic}, \tilde{P} \neq 0} \mathcal{D}\phi' e^{-S_E[\phi = \bar{\phi} + \phi']} \right\} \\ &\equiv \int_{-\infty}^{\infty} d\bar{\phi} \exp \left[-\frac{V}{T} V_{\text{eff}}(\bar{\phi}) \right]. \end{aligned} \quad (7.20)$$

Here ϕ' contains all modes with $\tilde{P} \neq 0$, and V_{eff} is called the (*constrained*) *effective potential*. The factor V/T is the trivial spacetime integral, $\int_0^\beta d\tau \int_V d^d \mathbf{x}$.

Let us write down the effective potential explicitly for the free theory, $\lambda = 0$. From Eqs. (7.15), (7.16), we get

$$S_E[\phi = \bar{\phi} + \phi'] = \frac{V}{T} (m^2 - \mu^2) \bar{\phi}^* \bar{\phi} + \sum_{\tilde{P}_b \neq 0} \tilde{\phi}^*(\tilde{P}) \tilde{\phi}(\tilde{P}) \left[(\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2 \right], \quad (7.21)$$

where we made use of the fact that the crossterm between $\bar{\phi}$ and ϕ' vanishes, given that by definition ϕ' has no zero-momentum mode:

$$\int_0^\beta d\tau \int_V d^d \mathbf{x} \phi' = 0. \quad (7.22)$$

Note that the latter term in Eq. (7.21) does *not* correspond to only non-zero Matsubara modes: Matsubara zero-modes ($\omega_n^b = 0$) with non-zero spatial momentum $\mathbf{p} \neq \mathbf{0}$ are also included. In the limit of a large volume, $mL \gg 1$, the omission of a single mode does not matter (its effect is $\propto (T/V) \ln(m^2 - \mu^2)$). The path integral over the latter term yields then Eq. (7.18), and in total we get

$$V_{\text{eff}}(\bar{\phi}) = (m^2 - \mu^2) \bar{\phi}^* \bar{\phi} + f(T, \mu). \quad (7.23)$$

Physically, the first term corresponds to the contribution to the free-energy density from particles that have formed a condensate, while the latter term represents free particles in the plasma.

Now, if we go toward zero temperature, $T \ll m$, and assume furthermore that $\mu \lesssim m$, then the latter term in Eq. (7.18) vanishes. The vacuum contribution left over is independent of μ , and can be omitted as well. Therefore

$$V_{\text{eff}}(\bar{\phi}) \simeq (m^2 - \mu^2) \bar{\phi}^* \bar{\phi}. \quad (7.24)$$

The remaining task is to carry out the final regular integral over $\bar{\phi}$ in Eq. (7.20). At this point we need to make contact with the particle number density. From the definition $\mathcal{Z} = \text{Tr} [\exp(-\beta\hat{H} + \beta\mu\hat{Q})]$, we obtain

$$\rho \equiv \frac{\langle \hat{Q} \rangle}{V} = \frac{T}{V} \frac{\partial \ln \mathcal{Z}}{\partial \mu} \quad (7.25)$$

$$= \frac{\int_{-\infty}^{\infty} d\bar{\phi} 2\mu \bar{\phi}^* \bar{\phi} \exp\left[-\frac{V}{T} V_{\text{eff}}(\bar{\phi})\right]}{\int_{-\infty}^{\infty} d\bar{\phi} \exp\left[-\frac{V}{T} V_{\text{eff}}(\bar{\phi})\right]}. \quad (7.26)$$

Let us now consider a situation where we decrease the temperature towards low values, $T \ll m$, and attempt simultaneously to keep the particle number density, the left-hand side of Eq. (7.26), fixed. How should we choose μ in this situation? There are three possibilities:

- (i) if $|\mu| > m$, the integrals are not defined (in the free theory). This simply means that such choices of μ are not physically meaningful.
- (ii) if $|\mu| < m$, the integrals can be carried out: in fact their result corresponds to the “propagator” of $\bar{\phi}$:

$$\rho = 2\mu \frac{T}{V(m^2 - \mu^2)}. \quad (7.27)$$

We note that if $T \rightarrow 0$, then $\rho \rightarrow 0$. This conflicts with our assumption that the particle number density is kept constant; therefore this range of μ is again not physically relevant for our situation.

- (iii) According to the preceding points, the only possible choice is $|\mu| = m$. More specifically, if $\rho > 0$, we need to choose $\mu = +m$. According to Eq. (7.27), this situation needs in fact to be approached by a careful tuning of $\mu \rightarrow m^-$ as we put $T \rightarrow 0$. However, assuming that this is done, and that we furthermore add an infinitesimal interaction $\lambda > 0$ to the theory so that the integrals in Eq. (7.26) are defined, we obtain the relation

$$\rho = 2m \langle \bar{\phi}^* \bar{\phi} \rangle > 0. \quad (7.28)$$

Eq. (7.28) manifests the phenomenon of *Bose-Einstein condensation* (at zero temperature in the free limit): the conserved particle number is converted to a non-zero scalar condensate.

Obviously, it would also be easy to include the effects of a finite temperature, by starting from Eq. (7.23), and the effects of interactions, by keeping $\lambda > 0$. These very interesting developments go beyond the scope of the present lectures, however. On the other hand, the concepts of a condensate and an effective potential will be met again in later chapters.