6.3. Dimensionally reduced effective field theory for hot QCD

We now apply the recipe of the previous section to the problem outlined in Sec. 6.1.

(1) Identification of soft degrees of freedom. As discussed in Sec. 6.1, the soft degrees of freedom are the bosonic Matsubara zero-modes. Since they do not depend on the coordinate τ, they live in $d = 3 - 2\epsilon$ spatial dimensions; thus the construction of the effective theory is in this context called *high-temperature dimensional reduction*[13].

(2) Symmetries. First of all, since the heat bath breaks Lorentz invariance, the time direction and the space directions are not symmetric. Indeed, the effective theory needs only to be invariant in *spatial rotations and translations.*

Second, the full theory possesses *discrete symmetries*; QCD is invariant in C, P and T separately. The effective theory inherits some reflection of these symmetries; in turns out, for instance, that $\mathcal{L}_{\rm eff}$ is symmetric in $\tilde{A}_0 \to -\tilde{A}_0$ (unless, say, C of QCD is broken by giving a chemical potential to the quarks).

Third, consider *gauge symmetries*, Eqs. (5.5), (5.6):

$$\tilde{A}'_\mu = U\tilde{A}_\mu U^{-1} + \frac{i}{g} U\partial_\mu U^{-1} \,. \tag{6.26}$$

Since we now restrict to *static* (i.e. τ-independent) fields, U (or θ^a) should not depend on τ either. Thus, the effective theory should be invariant under

$$\tilde{A}'_i = U\tilde{A}_i U^{-1} + \frac{i}{g} U\partial_i U^{-1} \,, \tag{6.27}$$

$$\tilde{A}'_0 = U\tilde{A}_0 U^{-1} \,. \tag{6.28}$$

In other words, the spatial components $\tilde{A}_i$ remain gauge fields, while the temporal components $\tilde{A}_0$ have turned into *scalar fields in the adjoint representation* (cf. Eq. (5.9)).

With these ingredients, we can write down the general form of the effective Lagrangian. It is illuminating to start by simply rewriting the full Lagrangian, Eq. (5.34), in terms of the soft degrees of freedom. Noting from Eq. (5.32),

$$F^a_{0i} \equiv \partial_\tau A^a_i - \mathcal{D}^{ab}_i \tilde{A}^b_0 \,, \tag{6.29}$$

that in the static case, $F^a_{i0} = \mathcal{D}^{ab}_i \tilde{A}^b_0$, we get

$$\mathcal{L}_E = \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} (\mathcal{D}^{ab}_i \tilde{A}^b_0)(\mathcal{D}^{ac}_i \tilde{A}^c_0) \,. \tag{6.30}$$

It is often convenient to note that

$$T^a \mathcal{D}^{ab}_i \tilde{A}^b_0 = \partial_i A_0 + g f^{acb} T^a A^c_i \tilde{A}^b_0 = \partial_i A_0 - ig[A_i, \tilde{A}_0] = [D_i, \tilde{A}_0] \,, \tag{6.31}$$

where $D_i = \partial_i - igA_i$ is the covariant derivative in the fundamental representation. Thereby,

$$\mathcal{L}^{(0)}_{\rm eff} = \frac{1}{4} \tilde{F}^a_{ij} \tilde{F}^a_{ij} + \mathrm{Tr}\,\{[D_i, \tilde{A}_0][D_i, \tilde{A}_0]\} \,. \tag{6.32}$$

[13]P. Ginsparg, *First and second order phase transitions in gauge theories at finite temperature,* Nucl. Phys. B 170 (1980) 388; T. Appelquist and R.D. Pisarski, *High-temperature Yang-Mills theories and three-dimensional Quantum Chromodynamics,* Phys. Rev. D 23 (1981) 2305.

Next, we complete the tree-level structure by adding all operators allowed by the symmetries. It is useful to proceed in order of increasing dimensionality. The following structures can be written down:

$$\text{dim} = 2: \qquad \text{Tr}\,[\tilde{A}_0^2]\;; \tag{6.33}$$

$$\text{dim} = 4: \qquad \text{Tr}\,[\tilde{A}_0^4]\;, \quad (\text{Tr}\,[\tilde{A}_0^2])^2\;; \tag{6.34}$$

$$\text{dim} = 6: \qquad \text{Tr}\,\{[D_i, F_{ij}][D_k, F_{kj}]\}\;, \ldots\;. \tag{6.35}$$

In the last case we have only shown one example; in total there is quite a large number of six-dimensional operators[14].

Combining Eqs. (6.32)–(6.34), we can write the effective action as

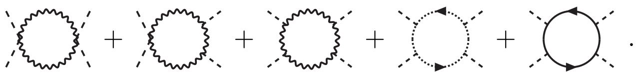
$$S_{\text{eff}} = \frac{1}{T}\int \mathrm{d}^d\mathbf{x}\left\{\frac{1}{4}\tilde{F}^a_{ij}\tilde{F}^a_{ij} + \text{Tr}\,([D_i,\tilde{A}_0][D_i,\tilde{A}_0]) + \tilde{m}^2\text{Tr}\,[\tilde{A}_0^2] + \tilde{\lambda}^{(1)}(\text{Tr}\,[\tilde{A}_0^2])^2 + \tilde{\lambda}^{(2)}\text{Tr}\,[\tilde{A}_0^4] + \ldots\right\}. \tag{6.36}$$

Here the prefactor $1/T$ comes from the integration $\int_0^\beta \mathrm{d}\tau$; since none of the soft fields depend on τ, we just get $1/T$, like in *classical* statistical physics. Sometimes this theory is referred to as EQCD, for "Electrostatic QCD"[15].

(3) Matching. If we restrict to 1-loop order, then the matching for the parameters in Eq. (6.36) is rather simple, as explained in Eq. (6.14): we simply need to compute Green's functions with soft fields (with vanishing external momenta) on the external legs, and heavy modes in the loop. For the parameter $\tilde{m}^2$, this is precisely the computation that we carried out in Sec. 5.5. Therefore, the result can be read directly from Eq. (5.96):

$$\tilde{m}^2 = g^2T^2\left(\frac{N_{\text{c}}}{3} + \frac{N_{\text{f}}}{6}\right) + \mathcal{O}(g^4T^2)\;. \tag{6.37}$$

The parameters $\tilde{\lambda}^{(1)}$, $\tilde{\lambda}^{(1)}$ can, in turn, be obtained by considering 4-point functions with soft modes of A_0 on the external legs, and non-zero Matsubara modes in the loop:



(6.38)

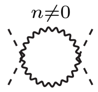
Clearly the effect is of $\mathcal{O}(g^4)$ and, using the same notation as in Eq. (5.96), the actual values are[16]

$$\tilde{\lambda}^{(1)} = \frac{g^4}{4\pi^2} + \mathcal{O}(g^6)\;, \quad \tilde{\lambda}^{(2)} = \frac{g^4}{12\pi^2}(N_{\text{c}} - N_{\text{f}}) + \mathcal{O}(g^6)\;. \tag{6.39}$$

(4) Truncation of higher dimensional operators. In some sense the most non-trivial and critical part of any effective field theory construction is the analysis of the accuracy that can be reached with the effective theory, once higher dimensional operators are dropped. In other words, the challenge is to determine the constant k in Eq. (6.11). Let us illustrate the procedure by considering the error that we make by dropping the operator in Eq. (6.35).

We need to know, first of all, the parametric magnitude of the coefficient with which the operator would enter $\mathcal{L}_{\text{eff}}$, if it were kept. This operator could be generated through the momentum dependence of graphs like

$n\neq 0$



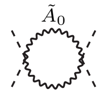
$$\sim \quad \frac{g^2}{T^2}(\partial_i F^a_{ij})^2\;, \tag{6.40}$$

[14] S. Chapman, *A new dimensionally reduced effective action for QCD at high temperature,* Phys. Rev. D 50 (1994) 5308.

[15] E. Braaten and A. Nieto, *Free energy of QCD at high temperature,* Phys. Rev. D 53 (1996) 3421.

[16] S. Nadkarni, *Dimensional reduction in finite temperature Quantum Chromodynamics. 2,* Phys. Rev. D 38 (1988) 3287; N.P. Landsman, *Limitations to dimensional reduction at high temperature,* Nucl. Phys. B 322 (1989) 498.

where the dashed lines stand for the spatial components A_i. If we now drop this term, the corresponding Green's function will not be computed correctly; however, it still has *some* value, namely that which would be obtained within the effective theory:



$$\sim \quad g^2(\partial_i F^a_{ij})^2 T \int_{\mathbf{p}} \frac{1}{(\mathbf{p}^2+\tilde{m}^2)^3} \sim \frac{g^2 T}{\tilde{m}^3}(\partial_i F^a_{ij})^2 \; . \tag{6.41}$$

Here we noted that there are two propagators, but to account for the dependence on the external momentum (represented by the derivative ∂_i in front of F^a_{ij}), we need to Taylor-expand in $\mathbf{p}$ to first non-trivial order.

We note from Eq. (6.41) that the Green's function is also non-zero within the effective theory; in fact, the contribution in Eq. (6.41) is *larger* than that in Eq. (6.40)! Therefore, the error made through the omission of Eq. (6.40) is *small*:

$$\frac{\delta\tilde{\Gamma}}{\tilde{\Gamma}} \sim \frac{g^2}{T^2}\frac{\tilde{m}^3}{g^2 T} \sim \left(\frac{\tilde{m}}{T}\right)^3 \sim g^3 \; . \tag{6.42}$$

In other words, for the dimensionally reduced effective theory of hot QCD, we get $k = 3$.[17]

Once the effective theory of Eq. (6.36) is there, we can take a further step: the field $\tilde{A}_0$ is massive, and can be integrated out. Thereby we arrive at an even simpler effective theory:

$$S'_{\text{eff}} = \frac{1}{T}\int \mathrm{d}^d\mathbf{x}\left\{\frac{1}{4}\tilde{\tilde{F}}{}^a_{ij}\tilde{\tilde{F}}{}^a_{ij} + \ldots\right\} . \tag{6.43}$$

Sometimes this theory is referred to as MQCD, for "Magnetostatic QCD"[15]. It is important to realise that this theory, the three-dimensional Yang-Mills theory (up to higher order operators such as the one in Eq. (6.35)), only has one parameter, the gauge coupling. Furthermore, if the fields $\tilde{\tilde{A}}{}^a_i$ are rescaled by an appropriate power of $T^{1/2}$, $\tilde{\tilde{A}}{}^a_i \to \tilde{\tilde{A}}{}^a_i T^{1/2}$, then the coefficient $1/T$ in Eq. (6.43) disappears. The coupling constant squared that appears afterwards is $\tilde{\tilde{g}}{}^2 T$, and this is the only scale in the system. Therefore all dimensionfull quantities (correlation lengths, string tension, free energy density, ...) must be proportional to an appropriate power of $\tilde{\tilde{g}}{}^2 T$, with a non-perturbative coefficient. This is the essence of the non-perturbative physics pointed out by Linde[10].

The implication of the complete setup for the weak-coupling expansion is the following. Consider a generic observable $\mathcal{O}$, with an expectation value

$$\langle\mathcal{O}\rangle \sim g^m T^n [1 + \# g^p + ...] \; . \tag{6.44}$$

It could now happen that: (i) p is even; # is determined by the heavy scale $\sim \pi T$, and is purely perturbative; (ii) p is even or odd; # is determined by the intermediate scale $\sim gT$, and is purely perturbative; (iii) $m+p$ is even; # is determined by the soft scale $\sim g^2 T$, and is non-perturbative; (iv) $p > k$; # can only be determined correctly by adding higher dimensional operators to the effective theory.

A final remark, relevant for effective field theories quite in general, is in order. Indeed, we have seen that the omission of higher order operators usually leads to a small error, since the same Green's function is produced with a larger coefficient within the effective field theory. It could happen, however, that there is some approximate symmetry in the full theory, which *becomes exact* within the effective theory, if we truncate to some order. For instance, many Grand Unified Theories induce violation of baryon minus lepton number $(B-L)$, while in the Standard Model this is an exact symmetry. It is then only broken by some higher dimensional operator[18]. Therefore, if we consider $B-L$ violation with the Standard Model, we make an *infinitely large* relative error.

[17] K. Kajantie, M. Laine, K. Rummukainen and M.E. Shaposhnikov, *High temperature dimensional reduction and parity violation,* Phys. Lett. B 423 (1998) 137.

[18] S. Weinberg, *Baryon and Lepton Nonconserving Processes*, Phys. Rev. Lett. 43 (1979) 1566; F. Wilczek and A. Zee, *Operator Analysis of Nucleon Decay*, Phys. Rev. Lett. 43 (1979) 1571.

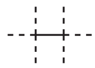
6.4. Exercise 9

Let us consider the full theory

$$\mathcal{L}_{\text{full}} \equiv \frac{1}{2}\,\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}\,m^2\phi^2 + \frac{1}{2}\,\partial_\mu H\partial_\mu H + \frac{1}{2}\,M^2H^2 + \frac{1}{6}\,\gamma H\phi^3\;. \tag{6.45}$$

For simplicity, we assume that the dimensionality of spacetime is 3 (i.e. $d = 2 - 2\epsilon$ in our standard notation). Moreover we work at zero temperature, like in Sec. 6.2.

(a) Integrating out H in order to construct an effective theory amounts to the computation of the graph



$$. \tag{6.46}$$

After Taylor-expanding in external momenta, write down all the corresponding operators.

(b) Let us consider the 4-point function of $\tilde{\phi}$'s at vanishing external momenta. What kind of contributions do the operators computed in part (a) give to this Green's function?

(c) Let us then consider directly the graph

$$\tag{6.47}$$

at vanishing external momenta. How does the result compare with the Taylor-expanded result of part (b)? What is the lesson?

Solution to Exercise 9

(a) The construction of the effective theory proceeds essentially as in Eq. (3.56), except that only the H-fields are integrated out. We get

$$\begin{aligned}
S_{\text{eff}} &\approx \Big\langle -\frac{1}{2}\,S_I^2 \Big\rangle_{H,\text{connected}} \\
&= -\frac{\gamma^2}{72}\int_{x,y}\phi^3(x)\phi^3(y)\langle H(x)H(y)\rangle_0 \\
&= -\frac{\gamma^2}{72}\int_{x,y}\phi^3(x)\phi^3(y)\int_p \frac{e^{ip\cdot(x-y)}}{p^2+M^2} \\
&= -\frac{\gamma^2}{72}\int_{x,y}\phi^3(x)\phi^3(y)\int_p e^{ip\cdot(x-y)}\Bigg[\sum_{n=0}^{\infty}\frac{(-1)^n(p^2)^n}{(M^2)^{n+1}}\Bigg] \\
&= -\frac{\gamma^2}{72}\int_{x,y}\phi^3(x)\phi^3(y)\Bigg[\sum_{n=0}^{\infty}\frac{(\nabla_x^2)^n}{(M^2)^{n+1}}\Bigg]\delta\!\!\!{}^-(x-y) \\
&= -\frac{\gamma^2}{72}\int_x\sum_{n=0}^{\infty}\phi^3(x)\frac{(\nabla_x^2)^n}{(M^2)^{n+1}}\phi^3(x)\;.
\end{aligned} \tag{6.48}$$

(b)

$$\begin{aligned}
&\Big\langle \tilde{\phi}(0)\tilde{\phi}(0)\tilde{\phi}(0)\tilde{\phi}(0)e^{-S_{\text{eff}}} \Big\rangle \\
&\Rightarrow\; \frac{\gamma^2}{72}\Big\langle \tilde{\phi}(0)\tilde{\phi}(0)\tilde{\phi}(0)\tilde{\phi}(0)\int_{P_1,\ldots,P_6}\delta\!\!\!{}^-(\Sigma_i P_i)\tilde{\phi}(P_1)\ldots\tilde{\phi}(P_6)\Big\rangle\sum_{n=0}^{\infty}\frac{[-(P_4+P_5+P_6)^2]^n}{(M^2)^{n+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\gamma^2}{72} \times 6 \times (2\times 3 \times 2 + 3 \times 4 \times 2) \int_{P_1,\ldots,P_6} \tilde{\delta}(\Sigma_i P_i) \sum_{n=0}^{\infty} \frac{[-(P_4+P_5+P_6)^2]^n}{(M^2)^{n+1}} \\
&\quad \times \langle \tilde{\phi}(0)\tilde{\phi}(P_1)\rangle \langle \tilde{\phi}(0)\tilde{\phi}(P_2)\rangle \langle \tilde{\phi}(0)\tilde{\phi}(P_5)\rangle \langle \tilde{\phi}(0)\tilde{\phi}(P_6)\rangle \langle \tilde{\phi}(P_3)\tilde{\phi}(P_4)\rangle \\
&= 3\gamma^2 \frac{\tilde{\delta}(0)}{(\tilde{m}^2)^4} \int_{P_3} \frac{1}{P_3^2+\tilde{m}^2} \sum_{n=0}^{\infty} \frac{(-P_3^2)^n}{(M^2)^{n+1}} \, . \qquad (6.49)
\end{aligned}
$$

The integrals in Eq. (6.49) can all be carried out in dimensional regularization; for instance, the two leading terms read

$$
n = 0 : \qquad \frac{1}{M^2} \int_{P_3} \frac{1}{P_3^2+\tilde{m}^2} = \frac{1}{M^2}\left(-\frac{\tilde{m}}{4\pi}\right) , \qquad (6.50)
$$

$$
n = 1 : \qquad -\frac{1}{M^4} \int_{P_3} \frac{P_3^2}{P_3^2+\tilde{m}^2} = \frac{\tilde{m}^2}{M^4} \int_{P_3} \frac{1}{P_3^2+\tilde{m}^2} = -\frac{1}{M^4}\frac{\tilde{m}^3}{4\pi} , \qquad (6.51)
$$

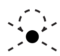
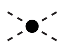
where we made use of Eq. (2.85). We note that, indeed, the terms get smaller with increasing n, apparently justifying *a posteriori* the Taylor-expansion that we carried out in part (a).

(c) Let us, on the other hand, carry out the integral corresponding to Eq. (6.47), without a Taylor expansion. The contractions remain as in part (b), and we simply need to replace the integral in Eq. (6.49) by

$$
\begin{aligned}
\int_{P_3} \frac{1}{P_3^2+\tilde{m}^2}\frac{1}{P_3^2+M^2} &= \int_{P_3} \frac{1}{M^2-\tilde{m}^2}\left[\frac{1}{P_3^2+\tilde{m}^2} - \frac{1}{P_3^2+M^2}\right] \\
&= \frac{1}{M^2-\tilde{m}^2}\left(\frac{-1}{4\pi}\right)(\tilde{m}-M) \\
&= \frac{1}{4\pi(M+\tilde{m})} \\
&= \frac{1}{4\pi M}\left(1 - \frac{\tilde{m}}{M} + \frac{\tilde{m}^2}{M^2} + \ldots\right) . \qquad (6.52)
\end{aligned}
$$

Comparing now Eqs. (6.50), (6.51) with Eq. (6.52), we note that by carrying out the Taylor-expansion, i.e. the naive matching to various effective parameters, too early, we missed the leading contribution, the dominant term in Eq. (6.52)! The largest term we found, Eq. (6.50), is only next-to-leading order in Eq. (6.52).

The reason for this problem is the same as in Eq. (6.21): it has to be taken into account that the light fields ϕ can also have large momenta P_3, $P_3 \sim M$, in which situation a Taylor-expansion of $1/(P_3^2+M^2)$ is not allowed. Rather, we have to view Eq. (6.47) in analogy with Eq. (6.21):

 $=$  $+$  . (6.53)

The first term here corresponds to a naive replacement of Eq. (6.46) by a momentum-independent 6-point vertex, times a dynamical effect from soft fields; indeed the result, Eq. (6.50), is non-analytic in the parameter $\tilde{m}^2$.

The second term, on the other hand, corresponds to a contribution from hard ϕ-modes to the effective 4-point vertex. The result, the leading term in Eq. (6.52), is indeed analytic in the parameter $\tilde{m}^2$.