6. Low-energy effective field theories

6.1. The infrared problem of thermal field theory

Let us consider the types of integrals that appear in perturbation theory. According to Eqs. (2.34), (4.58), each new loop order (corresponding to an additional loop momentum) produces one of

$$\sum\!\!\!\!\!\!\!\int_{\tilde{P}_\mathrm{b}} f(\omega_n^\mathrm{b}, \mathbf{p}) = \int_\mathbf{p} \left\{ \frac{1}{2} \int_{-\infty - i0^+}^{+\infty - i0^+} \frac{\mathrm{d}\omega}{2\pi} \, [f(\omega, \mathbf{p}) + f(-\omega, \mathbf{p})][1 + 2n_\mathrm{B}(i\omega)] \right\} , \tag{6.1}$$

$$\sum\!\!\!\!\!\!\!\int_{\tilde{P}_\mathrm{f}} f(\omega_n^\mathrm{f}, \mathbf{p}) = \int_\mathbf{p} \left\{ \frac{1}{2} \int_{-\infty - i0^+}^{+\infty - i0^+} \frac{\mathrm{d}\omega}{2\pi} \, [f(\omega, \mathbf{p}) + f(-\omega, \mathbf{p})][1 - 2n_\mathrm{F}(i\omega)] \right\} , \tag{6.2}$$

depending on whether the new line is bosonic or fermionic. The functions f here contain propagators, and possibly structures emerging from the vertices; in the simplest case, $f(\omega, \mathbf{p}) \sim 1/(\omega^2 + E^2)$, where we again denote $E \equiv \sqrt{\mathbf{p}^2 + m^2}$.

Now, the structures which are most important (i.e. yield the biggest contributions) are those where the functions f are largest. Let us inspect this in terms of both the left-hand and the right-hand sides of Eqs. (6.1), (6.2).

For bosons, the largest contribution on the left-hand side of Eq. (6.1) is associated with the *Matsubara zero-mode*, $\omega_n^\mathrm{b} = 0$; for $f(\omega, \mathbf{p}) \sim 1/(\omega^2 + E^2)$, this gives the term

$$\delta \left\{ T \sum_{\omega_n^\mathrm{b}} f \right\} \Bigg|_{\omega_n^\mathrm{b} = 0} \sim \frac{T}{E^2} . \tag{6.3}$$

In terms of the right-hand side, we have to close the contour in the lower half-plane, and the large term is associated with *Bose-Einstein enhancement* around the pole $\omega = -iE$:

$$\begin{aligned} \delta\{\ldots\} &\sim \frac{1}{2} \frac{-2\pi i}{2\pi} \frac{2}{-2iE} \Big[1 + 2n_\mathrm{B}(E) \Big] = \frac{1}{E} \left(\frac{1}{2} + \frac{1}{e^{E/T} - 1} \right) \\ &\approx \frac{1}{E} \left(\frac{1}{2} + \frac{1}{E/T + E^2/2T^2} + \ldots \right) = \frac{T}{E^2} + \mathcal{O}\Big(\frac{1}{T}\Big) . \end{aligned} \tag{6.4}$$

For fermions, there is no Matsubara zero-mode on the left-hand side of Eq. (6.2), so that the largest terms have the magnitude

$$\delta \left\{ T \sum_{\omega_n^\mathrm{f}} f \right\} \Bigg|_{\omega_n^\mathrm{f} = \pm \pi T} \sim \frac{T}{(\pi T)^2} \sim \frac{1}{T} . \tag{6.5}$$

Similarly, in terms of the right-hand side of Eq. (6.2), we can estimate

$$\begin{aligned} \delta\{\ldots\} &\sim \frac{1}{2} \frac{-2\pi i}{2\pi} \frac{2}{-2iE} \Big[1 - 2n_\mathrm{F}(E) \Big] = \frac{1}{E} \left(\frac{1}{2} - \frac{1}{e^{E/T} + 1} \right) \\ &\approx \frac{1}{E} \left(\frac{1}{2} - \frac{1}{2 + E/T} + \ldots \right) = \mathcal{O}\Big(\frac{1}{T}\Big) . \end{aligned} \tag{6.6}$$

Given these estimates, let us try to construct a dimensionless expansion parameter associated with the loop expansion. Apart from an additional propagator, each loop order also brings in an additional vertex (or vertices); let us denote the corresponding coupling by g^2, as would be the case in gauge theory. Morover, the Matsubara summation involves a factor T, so we can assume that the expansion parameter contains the combination g^2T. We now have to use the other scales in the problem to transform this into a dimensionless numbers. For the Matsubara zero-modes, Eq. (6.3) tells that we are allowed to use inverse powers of E or, after integration over the spatial momenta, inverse powers of m. Therefore, we can assume that for large temperatures, $T \gg m$, the largest possible expansion parameter is

$$\epsilon_{\rm b} \sim \frac{g^2 T}{m} \ . \tag{6.7}$$

For fermions, in contrast, Eq. (6.5) indicates that inverse powers of E or, after integration over spatial momenta, m, *cannot appear*; we are lead to the estimate

$$\epsilon_{\rm f} \sim \frac{g^2 T}{T} \sim g^2 \ . \tag{6.8}$$

Obviously, we expect also some typical loop factor like $1/(4\pi)^2$ to appear, but this has been omitted for simplicity.

Assuming that we work in the weak-coupling limit, $g^2 \ll 1$, we can thus conclude the following:

- *Fermions* appear to be purely *perturbative* in these computations.
- Bosonic *Matsubara zero-modes* appear to be purely *non-perturbative* for $m \to 0$.
- The *resummations* we saw in Exercise 5 for scalar field theory and in Sec. 5.5 for QCD produce an effective thermal mass, $m^2_{\rm eff} \sim g^2T^2$. Then we may expect the expansion parameter in Eq. (6.7) to become $\sim g^2T/gT = g$. In other words, a small expansion parameter exists in principle, but the structure of the weak-coupling series is peculariar, with odd powers of g, as we have seen before.
- As we found in Eq. (5.95), colour-magnetic fields do not develop a thermal mass at $\mathcal{O}(g^2T^2)$. This might still happen at higher orders, so we can say that $m_{\rm eff} \lesssim g^2T$. Thereby the expansion parameter in Eq. (6.7) reads $\epsilon_{\rm b} \gtrsim g^2T/g^2T = 1$. In other words, *colour-magnetic fields cannot be treated perturbatively*; this is known as the *infrared problem* (or Linde problem) of thermal gauge field theories[10].

The situation we observe here, namely that serious infrared problems exist, but that they are related to quite particular degrees of freedom, is common in field theory. Correspondingly there is also a generic tool, called the *effective field theory* approach, which allows to isolate the infrared problems to a simple Lagrangian, and treat them in this simplified setting. In fact, the concept of effective field theories is not restricted to finite-temperature physics, but applies even at zero temperature, if the system possesses a *scale hierarchy* (in the thermal context, the hierarchy is often expressed as $g^2T/\pi \ll gT \ll \pi T$, where the first scale refers to the non-perturbative one generated for the colour-magnetic fields). Given the generic nature of effective field theories, let us indeed start by discussing the basic idea in a simple zero temperature setting.

6.2. A simple example of an effective field theory

Let us consider a zero-temperature Lagrangian containing two different scalar fields, ϕ and H, with widely different masses, m and M, respectively[11]:

$$\mathcal{L}_{\rm full} \equiv \frac{1}{2}\,\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}\,m^2\phi^2 + \frac{1}{2}\,\partial_\mu H\partial_\mu H + \frac{1}{2}\,M^2H^2 + g^2\phi^2H^2 + \frac{1}{4}\,\lambda\phi^4 + \frac{1}{4}\,\kappa H^4 \ . \tag{6.9}$$

[10] A.D. Linde, *Infrared problem in thermodynamics of the Yang-Mills gas,* Phys. Lett. B 96 (1980) 289.
[11] This example and the related discussion follow closely those in the book *Renormalization*, by J.C. Collins.

We assume that there exists the scale hierarchy $m \ll M$ (or, to be more precise, $m_R \ll M_R$; we leave out the subscripts in the following). The effective theory now has the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\,\partial_\mu\tilde{\phi}\partial_\mu\tilde{\phi} + \frac{1}{2}\,\tilde{m}^2\tilde{\phi}^2 + \frac{1}{4}\,\tilde{\lambda}\tilde{\phi}^4 + \ldots\;, \tag{6.10}$$

where an infinite set of higher dimensional operators have been dropped out. Note that in principle, if we also wanted to describe gravity with these theories, we could add a "fundamental" cosmological constant Λ in $\mathcal{L}_{\text{full}}$, and an effective cosmological constant $\tilde{\Lambda}$ in $\mathcal{L}_{\text{eff}}$.

The statement concerning the effective description goes roughly as follows: Let us assume that $m \lesssim gM$, and consider external momenta $p \lesssim gM$. In addition, we assume that all the couplings are parametrically of the same order of magnitude, $\lambda \sim \kappa \sim g^2$. Then the one-particle-irreducible Green's functions $\tilde{\Gamma}_n$, computed within the effective theory, reproduce those of the full theory, Γ_n, with a relative error

$$\frac{\delta\tilde{\Gamma}_n}{\tilde{\Gamma}_n} \equiv \frac{|\tilde{\Gamma}_n - \Gamma_n|}{\tilde{\Gamma}_n} \lesssim \mathcal{O}(g^k)\;, \quad k > 0\;, \tag{6.11}$$

if $\tilde{m}^2$ and $\tilde{\lambda}$ are tuned suitably. (We assume the normalizations of the fields $\tilde{\phi}$ to be so chosen that the kinetic term in Eq. (6.10) has its canonical form.) The number k may depend on the dimensionality of spacetime; it may also depend on n although a universal lower bound should exist; the lower bound could be arbitrarily increased by adding suitable higher dimensional operators to $\mathcal{L}_{\text{eff}}$, and in the limit of infinitely many such operators the effective description becomes exact.

A weaker form of the effective theory statement, although already sufficiently strong and simultaneously one which may be easier to make precise (for instance, it is effectively implemented in the so-called non-perturbative Symanzik improvement program of lattice QCD[12]), is that Green's functions are matched only "on-shell", rather than for arbitrary external momenta.

The general effective theory statement is, as sometimes said, at the same time almost trivial, and yet extremely difficult to prove. We will not prove the statement here either, but let us try to get an impression on how it arises, operationally, by inspecting explicitly 2-point Green's functions.

To be more precise, let us inspect the inverse propagator of the light field ϕ. In the full theory, at 1-loop level, this reads

$G^{-1} =$

$$= \; p^2 + m^2 + \; \Pi_l^{(1)}(0; m^2) \; + \; \Pi_h^{(1)}(0; M^2)\;, \tag{6.12}$$

where the dashed line represent the light field; the solid line the heavy field; and the subscripts l, h stand for light and heavy, respectively. The first argument of the functions $\Pi_l^{(1)}$, $\Pi_h^{(1)}$ is the external momentum.

Within the effective theory, the same computation yields

$\tilde{G}^{-1} =$

$$= \; p^2 + \tilde{m}^2 + \; \tilde{\Pi}_l^{(1)}(0; \tilde{m}^2)\;. \tag{6.13}$$

The equivalence of all Green's functions at the on-shell point should also imply the equivalence of pole masses, i.e. the locations of the on-shell points. By *matching* Eqs. (6.12) and (6.13), we see that this can indeed be achieved provided that

$$\tilde{m}^2 = m^2 + \Pi_h^{(1)}(0; M^2) + \mathcal{O}(g^4)\;. \tag{6.14}$$

[12]K. Jansen *et al.*, *Non-perturbative renormalization of lattice QCD at all scales,* Phys. Lett. B 372 (1996) 275.

Note that within perturbative theory the matching needs to be carried out "order-by-order": $\tilde{\Pi}_l^{(1)}(0;\tilde{m}^2)$ is already of 1-loop order, so inside it $\tilde{\lambda}, \tilde{m}^2$ can be replaced by λ, m^2, respectively, given that the difference between $\tilde{\lambda}$ and λ as well as $\tilde{m}^2$ and m^2 is itself of 1-loop order, whereby $\tilde{\Pi}_l^{(1)}(0;\tilde{m}^2) = \Pi_l^{(1)}(0;m^2) + \mathcal{O}(g^4)$.

The situation becomes considerably more complicated once we go to the 2-loop level. Let us analyse various types of graphs that exist in the full theory, and try to understand how they could be matched with the simpler contributions existing within the effective theory.

First of all, there are graphs involving only the light fields,

 . (6.15)

These can be directly matched with the corresponding graphs within the effective theory computation; as above, the fact that different parameters appear in the propagators (and vertices) is a higher order effect.

Second, there are graphs which clearly account for the "insignificant higher order effects" that we omitted in the 1-loop matching, but would play a role once we go to the 2-loop level:

 $\Leftrightarrow \quad (\tilde{m}^2 - m^2)\dfrac{\partial \Pi_l^{(1)}(0;m^2)}{\partial m^2}$, (6.16)

 $\Leftrightarrow \quad (\tilde{\lambda} - \lambda)\dfrac{\partial \Pi_l^{(1)}(0;m^2)}{\partial \lambda}$. (6.17)

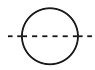
These two combine to reproduce (a part of) the 1-loop effective theory expression $\tilde{\Pi}_l^{(1)}(0;\tilde{m}^2)$ with 2-loop full theory accuracy.

Third, there are graphs only involving heavy fields in the loops:

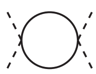
 . (6.18)

Obviously we can account for their effects by a 2-loop correction to $\tilde{m}^2$.

Finally, there remain the by far most complicated graphs: structures involving both heavy and light fields, in a way that the momenta flowing through the two sets of lines do *not* factorise:

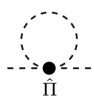
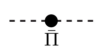
 =  . (6.19)

Naively, the representation on the right-hand side might suggest that this graph is simply part of the correction $(\tilde{\lambda} - \lambda)\partial \Pi_l^{(1)}(0;m^2)/\partial\lambda$, just like the graph in Eq. (6.17); this, however, is *not* the case, because the substructure appearing,

 , (6.20)

is momentum-dependent, unlike the effective vertex $\tilde{\lambda}$.

Nevertheless, it should be possible to split up Eq. (6.19) in two parts:

 $\equiv$  +  , (6.21)

$$\Pi_{mixed}^{(2)}(p^2;m^2,M^2) \equiv \hat{\Pi}_{mixed}^{(2)}(p^2;m^2,M^2) + \bar{\Pi}_{mixed}^{(2)}(p^2;m^2,M^2) \, . \tag{6.22}$$

The first part $\hat{\Pi}^{(2)}$ is, by definition, characterised by the fact that it depends *non-analytically* on the mass parameter m^2 of the light field; therefore the internal ϕ field is *soft* in this part, i.e. gets a contribution only from momenta $p \sim m$. In this situation, the momentum dependence of Eq. (6.20) is of subleading importance; indeed it can only be significant if the momentum flowing through the loop is of order $p \sim M$. In other words, this part of the graph *does* contribute simply to $(\tilde{\lambda} - \lambda)\partial\Pi_l^{(1)}(0; m^2)/\partial\lambda$, as we naively expected.

The second part $\bar{\Pi}^{(2)}$ is, by definition, analytic in the mass parameter m^2. We associate this with a situation where the internal ϕ is *hard*: even though its mass is small, it can have a large momentum $p \sim M$, transmitted to it through interactions with the heavy modes. In this situation, the momentum dependence of Eq. (6.20) plays an essential role. At the same time, the fact that all internal momenta are hard, allows for a Taylor-expansion in the small external momentum:

$$
\begin{aligned}
\bar{\Pi}^{(2)}_{mixed}(p^2; m^2, M^2) &= \bar{\Pi}^{(2)}_{mixed}(0; m^2, M^2) + p^2 \frac{\partial}{\partial p^2} \bar{\Pi}^{(2)}_{mixed}(0; m^2, M^2) \\
&\quad + \frac{1}{2}(p^2)^2 \frac{\partial^2}{\partial (p^2)^2} \bar{\Pi}^{(2)}_{mixed}(0; m^2, M^2) + \ldots \;. \qquad (6.23)
\end{aligned}
$$

The first term here represents now a 2-loop correction to $\tilde{m}^2$, just like the graph in Eq. (6.18). The second term can be compensated for by a change of the normalization of the field $\tilde{\phi}$. Finally, the further terms have the appearance of higher order (derivative) operators, truncated from the structure shown explicitly in Eq. (6.10). Compared with the leading kinetic term, the magnitude of the third term is very small, however:

$$
\frac{g^4 \frac{(p^2)^2}{M^2}}{p^2} \lesssim g^6 \;, \qquad (6.24)
$$

for $p \lesssim gM$. Therefore a truncation is indeed justified unless we want to go beyond a certain relative accuracy. The structures in Eq. (6.23) are collectively denoted by a 2-point "blob" in Eq. (6.21).

To summarise, we see that the explicit construction of the effective field theory becomes quite subtle at higher loop orders. Another illuminating example of the difficulties met with "mixed graphs" is given in Exercise 9.

Nevertheless, the following conjecture/philosophy/recipe for an effective field theory description can be formulated, and is assumed to hold in general (in weakly-coupled theories):

(1) Identify the "light" or "soft" degrees of freedom, i.e. the ones that are *IR-sensitive*.

(2) Write down the most general Lagrangian for them, respecting all the *symmetries* of the system, and including local operators of arbitrary order.

(3) The parameters of this Lagrangian can be determined by *matching*:

- compute the same observable in the full and in the effective theory, with the same UV-regularization and IR-cutoff.

- subtract the results.

- the IR-cutoff should disappear; the result of the subtraction should be analytic in p^2, and allow for a matching of the parameters and field normalizations of the effective theory.

- if not, the degrees of freedom, or the form of the effective theory, have not been correctly identified!

(4) *Truncate* the effective theory by dropping higher dimensional operators suppressed by $1/M^k$, which can only give a relative contribution of order

$$
\sim \left(\frac{m}{M}\right)^k \sim g^k \;. \qquad (6.25)
$$