5.6. Free energy density to $\mathcal{O}(g^3)$

As an immediate application of the results of the previous section, we compute the free energy density of QCD to $\mathcal{O}(g^3)$, parallelling the method introduced for scalar field theory in Exercise 5. We recall that the essential insight is to correct the quadratic part of the Lagrangian of the Matsubara zero modes by the effective thermal mass computed from the full propagator, and to treat minus the same term as part of the interaction Lagrangian. The "non-interacting" free energy density computed with the corrected propagator then yields the result for the ring sum, while the bilinear interaction term cancels the corresponding contributions order by order from multiloop graphs. In the present case, given the result in Eq. (5.95), only the zero-components of the gauge fields need to be corrected.

The correction of $\mathcal{O}(g^3)$ to the tree-level result[7] in Eq. (5.61) can therefore be immediately written down, by employing Eq. (2.86) and taking into account that there are N_c^2-1 gauge field components:

$$\begin{aligned} f_{(3/2)}(T)\big|_\mathrm{QCD} &= (N_\mathrm{c}^2-1)\Big(-\frac{Tm_\mathrm{E}^3}{12\pi}\Big) \\ &= (N_\mathrm{c}^2-1)T^4g^3\Big(-\frac{1}{12\pi}\Big)\Big(\frac{N_\mathrm{c}}{3}+\frac{N_\mathrm{f}}{6}\Big)^{\frac{3}{2}} \\ &= -\frac{\pi^2T^4}{90}(N_\mathrm{c}^2-1)g^3\frac{15}{2\pi^3}\Big(\frac{N_\mathrm{c}}{3}+\frac{N_\mathrm{f}}{6}\Big)^{\frac{3}{2}} \\ &= -\frac{\pi^2T^4}{90}60(N_\mathrm{c}^2-1)\Big(\frac{g^2}{4\pi^2}\Big)^{\frac{3}{2}}\Big(\frac{N_\mathrm{c}}{3}+\frac{N_\mathrm{f}}{6}\Big)^{\frac{3}{2}}\ , \end{aligned} \tag{5.97}$$

where the effective mass m_E was taken from Eq. (5.96).

Next, we need to consider the contributions of $\mathcal{O}(g^2)$. In analogy with Eq. (3.106), these terms[8] come from the non-zero mode contributions to 2-loop graphs:

$$f_{(1)}(T)\big|_\mathrm{QCD} = \Big\langle S_I - \frac{1}{2}\,S_I^2 + \ldots \Big\rangle_{0,\mathrm{connected,drop\ overall}\int_x}\ . \tag{5.98}$$

At this point, it is useful to compare Eq. (5.98) with the computation of the full propagator in the previous section, Eq. (5.94). We note that, apart from an overall minus sign, the two computations are quite similar; the main difference is in a combinatorial factor. In fact, the claim is that we only need to "close" the gluon line in the results of the previous section, and simultaneously divide the graph by $1/2n$, where n is the number of gluon lines in the vacuum graph. Let us prove this claim by direct inspection.

Consider first the contribution from the 4-gluon vertex. In vacuum graphs, this leads to the combinatorial factor

$$\langle A\,A\,A\,A\rangle_{0,\mathrm{c}} = 3\langle A\,A\rangle_0\,\langle A\,A\rangle_0\ , \tag{5.99}$$

while in the propagator we got

$$-\langle A\,A\;\;A\,A\,A\,A\rangle_{0,\mathrm{c}} = -4\times 3\langle A\,A\rangle_0\,\langle A\,A\rangle_0\,\langle A\,A\rangle_0\ , \tag{5.100}$$

so there is indeed a difference of $-4=-2\times n$, with $n=2$ being the number of contractions in Eq. (5.99). Similarly, with the contribution from two 3-gluon vertices, the vacuum graphs lead to the combinatorial factor

$$-\langle A\,A\,A\;\;A\,A\,A\rangle_{0,\mathrm{c}} = -3\times 2\langle A\,A\rangle_0\,\langle A\,A\rangle_0\,\langle A\,A\rangle_0\ , \tag{5.101}$$

[7] J.I. Kapusta, *Quantum Chromodynamics at High Temperature,* Nucl. Phys. B 148 (1979) 461.

[8] E.V. Shuryak, *Theory of Hadronic Plasma,* Sov. Phys. JETP 47 (1978) 212; S.A. Chin, *Transition to Hot Quark Matter in Relativistic Heavy Ion Collision,* Phys. Lett. B 78 (1978) 552.

while in the propagator we got

$$\langle A\,A\;\;A\,A\,A\;\;A\,A\,A\rangle_{0,\mathrm{c}} = 6\times 3\times 2\langle A\,A\rangle_0\;\langle A\,A\rangle_0\;\langle A\,A\rangle_0\;\langle A\,A\rangle_0\;, \tag{5.102}$$

so there is indeed a difference of $-6 = -2\times n$, with $n = 3$ the number of contractions in Eq. (5.101). Finally, the ghost and fermion contributions to the vacuum graphs lead to the combinatorial factor

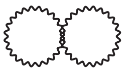
$$-\langle \bar{c}\,A\,c\;\bar{c}\,A\,c\rangle_{0,\mathrm{c}} = \langle A\,A\rangle_0\;\langle c\,\bar{c}\rangle_0\;\langle c\,\bar{c}\rangle_0\;, \tag{5.103}$$

while in the propagator we got

$$\langle A\,A\;\;\bar{c}\,A\,c\;\bar{c}\,A\,c\rangle_{0,\mathrm{c}} = -2\langle A\,A\rangle_0\;\langle A\,A\rangle_0\;\langle c\,\bar{c}\rangle_0\;\langle c\,\bar{c}\rangle_0\;, \tag{5.104}$$

so there is indeed a difference of $-2 = -2\times n$, with $n = 1$ the number of contractions in Eq. (5.103).

The contribution of the 4-gluon vertex can hence be extracted from Eq. (5.68):

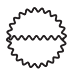


$$\begin{aligned} &= -\frac{1}{4}\left\{-g^2 N_\mathrm{c}\, d\, I_T^\mathrm{b}(0) \SumInt_{\tilde{P}_\mathrm{b}} \frac{\delta^{aa}\delta_{\mu\nu}}{\tilde{P}^2}\right\} \\ &= \frac{g^2}{4} N_\mathrm{c}(N_\mathrm{c}^2-1)d(d+1)[I_T^\mathrm{b}(0)]^2\;. \end{aligned} \tag{5.105}$$

The contribution of the 3-gluon vertices can be extracted from Eq. (5.73); noting from Eq. (5.74) that

$$\begin{aligned} &\delta_{\mu\nu}\; D_{\mu\beta\gamma}(-\tilde{P},\tilde{S},\tilde{P}-\tilde{S})\; D_{\nu\beta\gamma}(\tilde{P},-\tilde{S},-\tilde{P}+\tilde{S}) \\ &\quad= \; -(d+1)[4\tilde{P}^2 + (\tilde{P}-\tilde{S})^2 + \tilde{S}^2] - (d-5)\tilde{P}^2 + 2(2d-1)\tilde{P}\cdot\tilde{S} - (4d-2)\tilde{S}^2\;. \\ &\quad= \; -\{\tilde{P}^2[5d+5+d-5] + \tilde{P}\cdot\tilde{S}[-2d-2-4d+2] + \tilde{S}^2[2d+2+4d-2]\} \\ &\quad= \; -3d\{\tilde{P}^2 + (\tilde{P}-\tilde{S})^2 + \tilde{S}^2\}\;, \end{aligned} \tag{5.106}$$

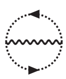
we get



$$\begin{aligned} &= -\frac{1}{6}\left\{\frac{3g^2N_\mathrm{c}}{2} d \SumInt_{\tilde{P}_\mathrm{b}} \frac{\delta^{aa}}{\tilde{P}^2} \SumInt_{\tilde{S}_\mathrm{b}} \frac{\tilde{P}^2 + (\tilde{P}-\tilde{S})^2 + \tilde{S}^2}{\tilde{S}^2(\tilde{P}-\tilde{S})^2}\right\} \\ &= -\frac{g^2}{4} N_\mathrm{c}(N_\mathrm{c}^2-1)d\times 3[I_T^\mathrm{b}(0)]^2\;. \end{aligned} \tag{5.107}$$

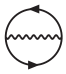
Note that here it is important to keep the full $\tilde{P}$-dependence, unlike in Eq. (5.75), because *all values of* $\tilde{P}$ are integrated over.

Similarly, the contribution of the ghost loop can be extracted from Eq. (5.84):



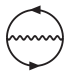
$$\begin{aligned} &= -\frac{1}{2}\left\{-g^2 N_\mathrm{c} \SumInt_{\tilde{P}_\mathrm{b}} \frac{\delta^{aa}}{\tilde{P}^2} \SumInt_{\tilde{S}_\mathrm{b}} \frac{\tilde{S}^2 - \tilde{P}\cdot\tilde{S}}{\tilde{S}^2(\tilde{P}-\tilde{S})^2}\right\} \\ &= \frac{g^2}{4} N_\mathrm{c}(N_\mathrm{c}^2-1) \SumInt_{\tilde{P}_\mathrm{b},\tilde{S}_\mathrm{b}} \frac{\tilde{S}^2 + (\tilde{P}-\tilde{S})^2 - \tilde{P}^2}{\tilde{P}^2\tilde{S}^2(\tilde{P}-\tilde{S})^2} \\ &= \frac{g^2}{4} N_\mathrm{c}(N_\mathrm{c}^2-1)[I_T^\mathrm{b}(0)]^2\;. \end{aligned} \tag{5.108}$$

Finally, the contribution of the fermion loop can be extracted from Eq. (5.90):



$$= -\frac{1}{2}\left\{2g^2 N_\mathrm{f} \SumInt_{\tilde{P}_\mathrm{b}} \frac{\delta^{aa}}{\tilde{P}^2} \SumInt_{\tilde{S}_\mathrm{f}} \frac{(d+1)(\tilde{S}^2 - \tilde{P}\cdot\tilde{S} + m^2) - 2\tilde{S}^2 + 2\tilde{P}\cdot\tilde{S}}{[\tilde{S}^2+m^2][(\tilde{P}-\tilde{S})^2+m^2]}\right\}\;. \tag{5.109}$$

We again simplify by setting $m/T \to 0$, and then get



$$
\begin{aligned}
&= -g^2 N_\mathrm{f}(N_\mathrm{c}^2-1) \sum\!\!\!\!\!\!\!\!\int_{\tilde{P}_\mathrm{b},\tilde{S}_\mathrm{f}} \frac{(d-1)(\tilde{S}^2 - \tilde{P}\cdot\tilde{S})}{\tilde{P}^2\tilde{S}^2(\tilde{P}-\tilde{S})^2} \\
&= -g^2 N_\mathrm{f}(N_\mathrm{c}^2-1) \frac{d-1}{2} \sum\!\!\!\!\!\!\!\!\int_{\tilde{P}_\mathrm{b},\tilde{S}_\mathrm{f}} \frac{\tilde{S}^2 + (\tilde{P}-\tilde{S})^2 - \tilde{P}^2}{\tilde{P}^2\tilde{S}^2(\tilde{P}-\tilde{S})^2} \\
&= -\frac{g^2}{2} N_\mathrm{f}(N_\mathrm{c}^2-1)(d-1)\{2I_T^\mathrm{b}(0)I_T^\mathrm{f}(0) - [I_T^\mathrm{f}(0)]^2\} \, . \qquad (5.110)
\end{aligned}
$$

In the last step, careful attention needs to be paid to the nature of Matsubara frequencies appearing in the propagators.

Adding together the terms from Eqs. (5.105), (5.107), (5.108), (5.110); setting $d = 3$ (there are no divergences); and using $I_T^\mathrm{b}(0) = T^2/12$, $I_T^\mathrm{f}(0) = -T^2/24$, we get

$$
\begin{aligned}
f_{(1)}(T)\big|_\mathrm{QCD} &= g^2(N_\mathrm{c}^2-1)\frac{T^4}{144}\left[\left(3 - \frac{9}{4} + \frac{1}{4}\right)N_\mathrm{c} - \left(-2\times\frac{1}{2} - \frac{1}{4}\right)N_\mathrm{f}\right] \\
&= g^2(N_\mathrm{c}^2-1)\frac{T^4}{144}\left(N_\mathrm{c} + \frac{5}{4}N_\mathrm{f}\right) \\
&= -\frac{\pi^2 T^4}{90}(N_\mathrm{c}^2-1)\left(-\frac{5}{2}\frac{g^2}{4\pi^2}\right)\left(N_\mathrm{c} + \frac{5}{4}N_\mathrm{f}\right) . \qquad (5.111)
\end{aligned}
$$

Combining finally the effects from Eqs. (5.61), (5.111), and (5.97), we get

$$
\begin{aligned}
f(T)\big|_\mathrm{QCD} &= -\frac{\pi^2 T^4}{45}(N_\mathrm{c}^2-1)\left\{1 + \frac{7}{4}\frac{N_\mathrm{f}N_\mathrm{c}}{N_\mathrm{c}^2-1} - \frac{5}{4}\left(N_\mathrm{c} + \frac{5}{4}N_\mathrm{f}\right)\frac{\alpha_s}{\pi} + \right. \\
&\qquad \left. +30\left(\frac{N_\mathrm{c}}{3} + \frac{N_\mathrm{f}}{6}\right)^{\frac{3}{2}}\left(\frac{\alpha_s}{\pi}\right)^{\frac{3}{2}} + \mathcal{O}(\alpha_s^2)\right\} , \qquad (5.112)
\end{aligned}
$$

were we have denoted $\alpha_s \equiv g^2/4\pi$.

A few remarks are in order:

- The result in Eq. (5.112) can be compared with that for scalar field theory in Eq. (3.100). The general structure is seen to be identical. In particular, in both cases, the first relative correction is *negative*. This means that the interactions between the particles in a plasma tend to *decrease* the pressure that the plasma exerts on the walls of an imaginary container.
- The second correction is positive, however. Such an alternating structure, even though in principle convergent (in the usual asymptotic sense) for small enough renormalized coupling, indicates that it may be difficult to quantitatively estimate the magnitude of radiative corrections to the non-interacting result. [But recall: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} ... = \ln 2 = 0.693...$, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} ... = \infty$; maybe an alternating structure is good after all!]
- As of today (fall 2007), the coefficients of the four subsequent terms, of orders $\mathcal{O}(\alpha_s^2 \ln\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^{5/2})$, and $\mathcal{O}(\alpha_s^3\ln\alpha_s)$, are also known.[9] Like for scalar field theory, this progress is possible via the use of effective field theory methods that we discuss in the next chapter.

[9] T. Toimela, *The next term in the thermodynamic potential of QCD,* Phys. Lett. B 124 (1983) 407; P. Arnold and C. Zhai, *The three loop free energy for pure gauge QCD,* Phys. Rev. D 50 (1994) 7603; C. Zhai and B. Kastening, *The free energy of hot gauge theories with fermions through* g^5*,* Phys. Rev. D 52 (1995) 7232; K. Kajantie, M. Laine, K. Rummukainen and Y. Schröder, *The pressure of hot QCD up to* $g^6\ln(1/g)$, Phys. Rev. D 67 (2003) 105008 [hep-ph/0211321].

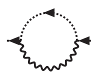
5.7. Exercise 8

In the computation in Sec. 5.6, we assumed that only the Matsubara zero modes of the fields A_0^a need to be resummed (i.e., get an effective thermal mass). We do know that fermions do not need to be resummed in any case, but how about ghots? Show that ghosts do *not* get any thermal mass, and thus behave like the spatial components of the gauge fields.

Solution to Exercise 8

The tree-level ghost propagator is in Eq. (5.46), and we now consider corrections to this expressions. The relevant vertex is the one in Eq. (5.50). We obtain

$$\Big\langle \tilde{c}^a(\tilde{P})\bar{\tilde{c}}^b(\tilde{Q})\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,\mathrm{c}}$$



$$\begin{aligned}
&= -\frac{g^2}{2}\Big\langle \tilde{c}^a(\tilde{P})\bar{\tilde{c}}^b(\tilde{Q})\sum\!\!\!\!\!\!\!\int_{\tilde{R}_\mathrm{b},\tilde{S}_\mathrm{b},\tilde{T}_\mathrm{b}} \bar{\tilde{c}}^c(\tilde{R})\tilde{A}^d_\alpha(\tilde{S})\tilde{c}^e(\tilde{T})\sum\!\!\!\!\!\!\!\int_{\tilde{U}_\mathrm{b},\tilde{V}_\mathrm{b},\tilde{X}_\mathrm{b}} \bar{\tilde{c}}^g(\tilde{U})\tilde{A}^h_\beta(\tilde{V})\tilde{c}^i(\tilde{X})\Big\rangle_{0,\mathrm{c}} \times \\
&\qquad \times f^{cde}f^{ghi}\delta(-\tilde{R}+\tilde{S}+\tilde{T})\delta(-\tilde{U}+\tilde{V}+\tilde{X})\;\tilde{R}_\alpha\tilde{U}_\beta \\
&= -g^2\sum\!\!\!\!\!\!\!\int_{\tilde{R}_\mathrm{b},\tilde{S}_\mathrm{b},\tilde{T}_\mathrm{b},\tilde{U}_\mathrm{b},\tilde{V}_\mathrm{b},\tilde{X}_\mathrm{b}} \langle \tilde{c}^a(\tilde{P})\bar{\tilde{c}}^c(\tilde{R})\rangle_0\,\langle \tilde{c}^e(\tilde{T})\bar{\tilde{c}}^g(\tilde{U})\rangle_0\,\langle \tilde{c}^i(\tilde{X})\bar{\tilde{c}}^b(\tilde{Q})\rangle_0\,\langle \tilde{A}^d_\alpha(\tilde{S})\tilde{A}^h_\beta(\tilde{V})\rangle_0 \;\times \\
&\qquad \times f^{cde}f^{ghi}\delta(-\tilde{R}+\tilde{S}+\tilde{T})\delta(-\tilde{U}+\tilde{V}+\tilde{X})\;\tilde{R}_\alpha\tilde{U}_\beta\;. \qquad (5.113)
\end{aligned}$$

Inserting the gluon propagator from Eq. (5.63) and the free ghost propagator from Eq. (5.46), let us inspect in turn colour indices, Lorentz indices, and momenta. The colour contractions result in a factor

$$\delta^{ac}\delta^{eg}\delta^{ib}\delta^{dh}f^{cde}f^{ghi} = f^{ade}f^{edb} = -N_\mathrm{c}\delta^{ab}\;. \qquad (5.114)$$

The spacetime indices yield simply $\delta_{\alpha\beta}$. The momentum dependence can be written as

$$\begin{aligned}
&\frac{\delta(\tilde{P}-\tilde{R})\delta(\tilde{T}-\tilde{U})\delta(\tilde{X}-\tilde{Q})\delta(\tilde{S}+\tilde{V})}{\tilde{P}^2\tilde{T}^2\tilde{Q}^2\tilde{S}^2}\delta(-\tilde{R}+\tilde{S}+\tilde{T})\delta(-\tilde{U}+\tilde{V}+\tilde{X})\;\tilde{R}\cdot\tilde{U} \\
=\;&\frac{\delta(\tilde{P}-\tilde{R})\delta(\tilde{T}-\tilde{U})\delta(\tilde{X}-\tilde{Q})\delta(\tilde{S}+\tilde{V})}{\tilde{P}^2\tilde{T}^2\tilde{Q}^2\tilde{S}^2}\delta(-\tilde{P}+\tilde{S}+\tilde{T})\delta(-\tilde{T}-\tilde{S}+\tilde{Q})\;\tilde{P}\cdot\tilde{T} \\
=\;&\frac{\delta(\tilde{P}-\tilde{R})\delta(\tilde{T}-\tilde{U})\delta(\tilde{Q}-\tilde{X})\delta(\tilde{S}+\tilde{V})}{\tilde{P}^2\tilde{Q}^2\tilde{T}^2(\tilde{P}-\tilde{T})^2}\delta(-\tilde{P}+\tilde{S}+\tilde{T})\delta(-\tilde{P}+\tilde{Q})\;\tilde{P}\cdot\tilde{T}\;. \qquad (5.115)
\end{aligned}$$

We can now integrate over $\tilde{R},\tilde{U},\tilde{X},\tilde{V}$ and $\tilde{S}$. Thereby

$$\Big\langle \tilde{c}^a(\tilde{P})\bar{\tilde{c}}^b(\tilde{Q})\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,\mathrm{c}} = g^2N_\mathrm{c}\frac{\delta^{ab}\delta(\tilde{P}-\tilde{Q})}{(\tilde{P}^2)^2}\sum\!\!\!\!\!\!\!\int_{\tilde{S}_\mathrm{b}}\frac{\tilde{P}\cdot\tilde{S}}{\tilde{S}^2(\tilde{P}-\tilde{S})^2}\;, \qquad (5.116)$$

where we renamed $\tilde{T}\to\tilde{S}$.

We note that the result here is proportional to the external momentum $\tilde{P}$. Therefore, the result in Eq. (5.116) does not represent an effective mass correction; it is rather a wave function (re)normalization contribution.