5.5. Thermal gluon mass

Consider the propagator of the Matsubara zero modes of gauge fields. We wish to see whether an effective thermal mass $m_{\rm eff}$ is generated for them, like for the scalar fields (cf. Eq. (3.102)). (Note that we do not need to consider non-zero Matsubara modes since, like in Eq. (3.101), the thermal mass corrections are subdominant in this case, $g^2T^2 \ll (2\pi T)^2$. For the same reason, we do not need to consider thermal mass correction for fermions at the present order.) The observable to consider is the full propagator, the analogue of Eq. (3.71).

In order to simplify the task somewhat, we choose to carry out the computation in the so-called Feynman gauge, $\xi \equiv 1$. Then the free propagator from Eq. (5.45) becomes

$$\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Big\rangle_0 \; = \; \delta^{ab}\,\bar\delta(\tilde P+\tilde Q)\bigg[\frac{\delta_{\mu\nu}-\frac{\tilde P_\mu \tilde P_\nu}{\tilde P^2}}{\tilde P^2}+\frac{\xi\frac{\tilde P_\mu \tilde P_\nu}{\tilde P^2}}{\tilde P^2}\bigg] \; \overset{\xi=1}{=} \; \delta^{ab}\,\bar\delta(\tilde P+\tilde Q)\frac{\delta_{\mu\nu}}{\tilde P^2}\,. \qquad (5.63)$$

At 1-loop level, in QCD, there are corrections to Eq. (5.63) from two graphs involving gauge loops (via a quartic vertex in $-S_I$ and via cubic vertices in $+S_I^2/2$); from one graph involving ghosts (via cubic vertices in $+S_I^2/2$); and from one graph involving fermions (via cubic vertices in $+S_I^2/2$). (If the theory contains scalar fields, there are two graphs like from gauge loops; for simplicity, we omit this contribution in the following.)

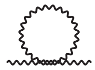
For future reference, we will treat $\tilde P$ as a general Euclidean four-momentum for the moment, even though for the Matsubara zero modes only the spatial part is non-zero.

Let us start by considering the gauge loop via a quartic vertex. Denoting the structure in Eq. (5.49) by

$$C^{cdef}_{\alpha\beta\sigma\rho} \equiv f^{gcd}f^{gef}(\delta_{\alpha\rho}\delta_{\beta\sigma}-\delta_{\alpha\sigma}\delta_{\beta\rho}) + f^{gce}f^{gdf}(\delta_{\alpha\beta}\delta_{\rho\sigma}-\delta_{\alpha\sigma}\delta_{\beta\rho}) + f^{gcf}f^{gde}(\delta_{\alpha\beta}\delta_{\rho\sigma}-\delta_{\alpha\rho}\delta_{\beta\sigma})\,, \qquad (5.64)$$

we get

$$\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)(-S_I)\Big\rangle_{0,\rm c}$$



$$= -\frac{g^2}{24}\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q) \SumInt_{\tilde R_{\rm b},\tilde S_{\rm b},\tilde T_{\rm b},\tilde U_{\rm b}} \tilde A^c_\alpha(\tilde R)\tilde A^d_\beta(\tilde S)\tilde A^e_\rho(\tilde T)\tilde A^f_\sigma(\tilde U)\,\bar\delta(\tilde R+\tilde S+\tilde T+\tilde U)\; C^{cdef}_{\alpha\beta\rho\sigma}\Big\rangle_{0,\rm c}$$

$$= -\frac{g^2}{2}\SumInt_{\tilde R_{\rm b},\tilde S_{\rm b},\tilde T_{\rm b},\tilde U_{\rm b}} \bar\delta(\tilde R+\tilde S+\tilde T+\tilde U)\langle \tilde A^a_\mu(\tilde P)\tilde A^c_\alpha(\tilde R)\rangle_0\; \langle \tilde A^b_\nu(\tilde Q)\tilde A^d_\beta(\tilde S)\rangle_0\; \langle \tilde A^e_\rho(\tilde T)\tilde A^f_\sigma(\tilde U)\rangle_0\; C^{cdef}_{\alpha\beta\rho\sigma}\,, \qquad (5.65)$$

where we made use of the complete symmetry of $C^{cdef}_{\alpha\beta\rho\sigma}$. Inserting Eq. (5.63), we get

$$\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)(-S_I)\Big\rangle_{0,\rm c} \; = \; -\frac{g^2}{2}\SumInt_{\tilde R_{\rm b},\tilde S_{\rm b},\tilde T_{\rm b},\tilde U_{\rm b}} \bar\delta(\tilde R+\tilde S+\tilde T+\tilde U)\bar\delta(\tilde P+\tilde R)\bar\delta(\tilde Q+\tilde S)\bar\delta(\tilde T+\tilde U)\;\times$$

$$\times\frac{1}{\tilde P^2\tilde Q^2\tilde T^2}\delta^{ac}\delta^{bd}\delta^{ef}\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\rho\sigma}C^{cdef}_{\alpha\beta\rho\sigma}\,. \qquad (5.66)$$

The sum-integrals over $\tilde R, \tilde S, \tilde U$ are trivially carried out. Moreover, we note that

$$\delta^{ac}\delta^{bd}\delta^{ef}\delta_{\mu\alpha}\delta_{\nu\beta}\delta_{\rho\sigma}C^{cdef}_{\alpha\beta\rho\sigma} \; = \; \delta^{ef}\delta_{\rho\sigma}\Big[f^{gae}f^{gbf}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}) + f^{gaf}f^{gbe}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\rho}\delta_{\nu\sigma})\Big]$$

$$= \; 2\,d\,f^{age}f^{bge}\delta_{\mu\nu}\,, \qquad (5.67)$$

where we made use of the antisymmetry of the structure constants, and the fact that $\delta_{\sigma\sigma} = d+1$ (which in dimensional regularization equals $4-2\epsilon$). The structure constants satisfy $f^{age}f^{bge} =$

$N_c\delta^{ab}$, and in total we then get

$$\Big\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})(-S_I)\Big\rangle_{0,\mathrm{c}} = -g^2 N_\mathrm{c}\, d\, I^\mathrm{b}_T(0)\, \delta^{ab}\delta(\tilde{P}+\tilde{Q})\frac{\delta_{\mu\nu}}{(\tilde{P}^2)^2}\ . \tag{5.68}$$

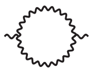
Note that the structure in the colour and spacetime indices is the same as in Eq. (5.63).

We then move on to the gluon loop originating from the cubic interactions. Denoting the combination in Eq. (5.48) by

$$D_{\alpha\beta\gamma}(\tilde{R},\tilde{S},\tilde{T}) \equiv \delta_{\alpha\gamma}(\tilde{R}_\beta - \tilde{T}_\beta) + \delta_{\gamma\beta}(\tilde{T}_\alpha - \tilde{S}_\alpha) + \delta_{\beta\alpha}(\tilde{S}_\gamma - \tilde{R}_\gamma)\ , \tag{5.69}$$

we get

$$\Big\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\Big(\frac{1}{2}S_I^2\Big)\Big\rangle_{0,\mathrm{c}}$$



$$\begin{aligned}
&= -\frac{g^2}{72}\Big\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q}) \sum\!\!\!\!\!\!\!\int_{\tilde{R}_\mathrm{b},\tilde{S}_\mathrm{b},\tilde{T}_\mathrm{b}} \tilde{A}^c_\alpha(\tilde{R})\tilde{A}^d_\beta(\tilde{S})\tilde{A}^e_\gamma(\tilde{T}) \sum\!\!\!\!\!\!\!\int_{\tilde{U}_\mathrm{b},\tilde{V}_\mathrm{b},\tilde{X}_\mathrm{b}} \tilde{A}^g_\zeta(\tilde{U})\tilde{A}^h_\eta(\tilde{V})\tilde{A}^i_\rho(\tilde{X})\Big\rangle_{0,\mathrm{c}} \times \\
&\quad\times f^{cde}f^{ghi}\delta(\tilde{R}+\tilde{S}+\tilde{T})\delta(\tilde{U}+\tilde{V}+\tilde{X})\ D_{\alpha\beta\gamma}(\tilde{R},\tilde{S},\tilde{T})\ D_{\zeta\eta\rho}(\tilde{U},\tilde{V},\tilde{X}) \\
&= -\frac{g^2}{2} \sum\!\!\!\!\!\!\!\int_{\tilde{R}_\mathrm{b},\tilde{S}_\mathrm{b},\tilde{T}_\mathrm{b},\tilde{U}_\mathrm{b},\tilde{V}_\mathrm{b},\tilde{X}_\mathrm{b}} \langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^c_\alpha(\tilde{R})\rangle_0\ \langle \tilde{A}^b_\nu(\tilde{Q})\tilde{A}^g_\zeta(\tilde{U})\rangle_0\ \langle \tilde{A}^d_\beta(\tilde{S})\tilde{A}^h_\eta(\tilde{V})\rangle_0\ \langle \tilde{A}^e_\gamma(\tilde{T})\tilde{A}^i_\rho(\tilde{X})\rangle_0 \times \\
&\quad\times f^{cde}f^{ghi}\delta(\tilde{R}+\tilde{S}+\tilde{T})\delta(\tilde{U}+\tilde{V}+\tilde{X})\ D_{\alpha\beta\gamma}(\tilde{R},\tilde{S},\tilde{T})\ D_{\zeta\eta\rho}(\tilde{U},\tilde{V},\tilde{X})\ ,
\end{aligned} \tag{5.70}$$

where we made use of the complete symmetry of $f^{cde}D_{\alpha\beta\gamma}(\tilde{R},\tilde{S},\tilde{T})$ in simultaneous interchanges of all indices labelling a particular gauge field (for instance $c,\alpha,\tilde{R} \leftrightarrow d,\beta,\tilde{S}$).

Inserting Eq. (5.63), let us inspect in turn colour indices, spacetime indices, and momenta. The colour contractions result in a factor

$$\delta^{ac}\delta^{bg}\delta^{dh}\delta^{ei}f^{cde}f^{ghi} = f^{ade}f^{bde} = N_\mathrm{c}\delta^{ab}\ . \tag{5.71}$$

The spacetime indices can be transported to the D-functions: $\alpha\to\mu$, $\zeta\to\nu$, $\eta\to\beta$, $\rho\to\gamma$. The momentum dependence can be written as

$$\begin{aligned}
&\frac{\delta(\tilde{P}+\tilde{R})\delta(\tilde{Q}+\tilde{U})\delta(\tilde{S}+\tilde{V})\delta(\tilde{T}+\tilde{X})}{\tilde{P}^2\tilde{Q}^2\tilde{S}^2\tilde{T}^2}\times \\
&\qquad\times\delta(\tilde{R}+\tilde{S}+\tilde{T})\delta(\tilde{U}+\tilde{V}+\tilde{X})\ D_{\mu\beta\gamma}(\tilde{R},\tilde{S},\tilde{T})\ D_{\nu\beta\gamma}(\tilde{U},\tilde{V},\tilde{X}) \\
=\ &\frac{\delta(\tilde{P}+\tilde{R})\delta(\tilde{Q}+\tilde{U})\delta(\tilde{S}+\tilde{V})\delta(\tilde{T}+\tilde{X})}{\tilde{P}^2\tilde{Q}^2\tilde{S}^2\tilde{T}^2}\times \\
&\qquad\times\delta(-\tilde{P}+\tilde{S}+\tilde{T})\delta(-\tilde{Q}-\tilde{S}-\tilde{T})\ D_{\mu\beta\gamma}(-\tilde{P},\tilde{S},\tilde{T})\ D_{\nu\beta\gamma}(-\tilde{Q},-\tilde{S},-\tilde{T}) \\
=\ &\frac{\delta(\tilde{P}+\tilde{R})\delta(\tilde{Q}+\tilde{U})\delta(\tilde{S}+\tilde{V})\delta(\tilde{T}+\tilde{X})}{\tilde{P}^2\tilde{Q}^2\tilde{S}^2\tilde{T}^2}\times \\
&\qquad\times\delta(-\tilde{P}+\tilde{S}+\tilde{T})\delta(\tilde{P}+\tilde{Q})\ D_{\mu\beta\gamma}(-\tilde{P},\tilde{S},\tilde{T})\ D_{\nu\beta\gamma}(\tilde{P},-\tilde{S},-\tilde{T})
\end{aligned} \tag{5.72}$$

We can now integrate over $\tilde{R},\tilde{U},\tilde{V},\tilde{X}$ and $\tilde{T}$. Thereby

$$\begin{aligned}
&\Big\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\Big(\frac{1}{2}S_I^2\Big)\Big\rangle_{0,\mathrm{c}} \\
&= -\frac{g^2 N_\mathrm{c}}{2}\frac{\delta^{ab}\delta(\tilde{P}+\tilde{Q})}{(\tilde{P}^2)^2} \sum\!\!\!\!\!\!\!\int_{\tilde{S}_\mathrm{b}} \frac{1}{\tilde{S}^2(\tilde{P}-\tilde{S})^2}\ D_{\mu\beta\gamma}(-\tilde{P},\tilde{S},\tilde{P}-\tilde{S})\ D_{\nu\beta\gamma}(\tilde{P},-\tilde{S},-\tilde{P}+\tilde{S})\ .
\end{aligned} \tag{5.73}$$

Finally, we are faced with the tedious task of inserting Eq. (5.69) and carrying out the contractions:

$$D_{\mu\beta\gamma}(-\tilde{P},\tilde{S},\tilde{P}-\tilde{S})\ D_{\nu\beta\gamma}(\tilde{P},-\tilde{S},-\tilde{P}+\tilde{S})$$

$$
\begin{aligned}
&= -[\delta_{\mu\gamma}(-2\tilde{P}_\beta+\tilde{S}_\beta)+\delta_{\gamma\beta}(\tilde{P}_\mu-2\tilde{S}_\mu)+\delta_{\beta\mu}(\tilde{P}_\gamma+\tilde{S}_\gamma)]\times \\
&\quad \times[\delta_{\nu\gamma}(-2\tilde{P}_\beta+\tilde{S}_\beta)+\delta_{\gamma\beta}(\tilde{P}_\nu-2\tilde{S}_\nu)+\delta_{\beta\nu}(\tilde{P}_\gamma+\tilde{S}_\gamma)] \\
&= -\delta_{\mu\nu}(4\tilde{P}^2-4\tilde{P}\cdot\tilde{S}+\tilde{S}^2+\tilde{P}^2+2\tilde{P}\cdot\tilde{S}+\tilde{S}^2)-(d+1)(\tilde{P}_\mu\tilde{P}_\nu-2\tilde{P}_\mu\tilde{S}_\nu-2\tilde{P}_\nu\tilde{S}_\mu+4\tilde{S}_\mu\tilde{S}_\nu) \\
&- \ [(-2\tilde{P}_\mu+\tilde{S}_\mu)(\tilde{P}_\nu-2\tilde{S}_\nu)+(-2\tilde{P}_\mu+\tilde{S}_\mu)(\tilde{P}_\nu+\tilde{S}_\nu)+(\tilde{P}_\mu-2\tilde{S}_\mu)(\tilde{P}_\nu+\tilde{S}_\nu)+(\mu\leftrightarrow\nu)] \\
&= -\delta_{\mu\nu}[4\tilde{P}^2+(\tilde{P}-\tilde{S})^2+\tilde{S}^2]-(d-5)\tilde{P}_\mu\tilde{P}_\nu+(2d-1)(\tilde{P}_\mu\tilde{S}_\nu+\tilde{P}_\nu\tilde{S}_\mu)-(4d-2)\tilde{S}_\mu\tilde{S}_\nu\ .
\end{aligned}
\tag{5.74}
$$

Inserting Eq. (5.74) into Eq. (5.73), we observe that the result depends in a non-trivial way on the "external" momentum $\tilde{P}$. This is an important fact which will play a role later on. Nevertheless, for the moment we may note that since the tree-level gluon propagator, Eq. (5.63), is massless, the pole position lies at $\tilde{P}^2=0$. This pole position may get shifted by the correction in Eq. (5.73), like happened in scalar field theory (cf. Eq. (3.102)), but since the correction is "small" (suppressed by the coupling), we may in fact insert $\tilde{P}^2=0$ inside the already small loop correction in Eq. (5.74), thereby only making an error of $\mathcal{O}(g^4)$. Proceeding this way, we get

$$
\Big\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,\mathrm{c}} \approx \frac{g^2N_\mathrm{c}}{2}\frac{\delta^{ab}\,\bar{\delta}(\tilde{P}+\tilde{Q})}{(\tilde{P}^2)^2}\SumInt_{\tilde{S}_\mathrm{b}}\frac{1}{(\tilde{S}^2)^2}\Big[2\tilde{S}^2\delta_{\mu\nu}+(4d-2)\tilde{S}_\mu\tilde{S}_\nu\Big]\ . \tag{5.75}
$$

Now, symmetries tell that the latter integral in Eq. (5.75) must be proportional to $\delta_{\mu\nu}$, just like the first one. However, its value could still be different for $\mu=\nu=0$ and $\mu=\nu=i$. This is because the heat bath constitutes a preferred frame, and thus breaks Lorentz invariance. In fact, we can write

$$
\begin{aligned}
\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}_\mu\tilde{S}_\nu}{(\tilde{S}^2)^2} &= \delta_{\mu0}\delta_{\nu0}\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}_0^2}{(\tilde{S}^2)^2}+\delta_{\mu i}\delta_{\nu i}\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}_i^2}{(\tilde{S}^2)^2} \\
&= \delta_{\mu0}\delta_{\nu0}\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}_0^2}{(\tilde{S}^2)^2}+\delta_{\mu i}\delta_{\nu i}\,\frac{1}{d}\SumInt_{\tilde{S}_\mathrm{b}}\frac{\mathbf{s}^2}{(\tilde{S}^2)^2} \\
&= \delta_{\mu0}\delta_{\nu0}\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}_0^2}{(\tilde{S}^2)^2}+\delta_{\mu i}\delta_{\nu i}\,\frac{1}{d}\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}^2-\tilde{S}_0^2}{(\tilde{S}^2)^2}\ .
\end{aligned}
\tag{5.76}
$$

At this point, let us inspect the sum-integral

$$
I_T^\mathrm{b}(0)=\SumInt_{\tilde{S}_\mathrm{b}}\frac{1}{\tilde{S}^2}=T\sum_{n=-\infty}^{\infty}\int_\mathbf{s}\frac{1}{(2\pi nT)^2+\mathbf{s}^2}=\frac{T^2}{12}+\mathcal{O}(\epsilon)\ . \tag{5.77}
$$

Taking the derivative $T^2\mathrm{d}/\mathrm{d}T^2=T/2\ \mathrm{d}/\mathrm{d}T$ on both sides, we find

$$
\frac{1}{2}\,T\sum_{n=-\infty}^{\infty}\int_\mathbf{s}\frac{1}{(2\pi nT)^2+\mathbf{s}^2}-T\sum_{n=-\infty}^{\infty}\int_\mathbf{s}\frac{(2\pi nT)^2}{[(2\pi nT)^2+\mathbf{s}^2]^2}=\frac{T^2}{12}+\mathcal{O}(\epsilon)\ , \tag{5.78}
$$

which can be used to solve for the unknown sum-integral,

$$
\SumInt_{\tilde{S}_\mathrm{b}}\frac{\tilde{S}_0^2}{(\tilde{S}^2)^2}=-\frac{1}{2}\,I_T^\mathrm{b}(0)+\mathcal{O}(\epsilon)\ . \tag{5.79}
$$

Inserting Eqs. (5.76), (5.79) into Eq. (5.75) finally yields

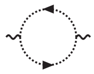
$$
\Big\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,\mathrm{c}} \approx \frac{g^2N_\mathrm{c}}{2}\frac{\delta^{ab}\,\bar{\delta}(\tilde{P}+\tilde{Q})}{(\tilde{P}^2)^2}\Big\{\delta_{\mu0}\delta_{\nu0}\Big[2-\frac{1}{2}\,(4d-2)\Big]+
$$

$$
\begin{aligned}
&\qquad + \delta_{\mu i}\delta_{\nu i}\bigg[2 + \frac{4d-2}{d}\times\frac{3}{2}\bigg]\bigg\} I_T^{\rm b}(0) \\
&= \frac{g^2 N_{\rm c}}{2}\frac{\delta^{ab}\delta(\tilde P + \tilde Q)}{(\tilde P^2)^2}\bigg\{-3\,\delta_{\mu 0}\delta_{\nu 0} + 7\,\delta_{\mu i}\delta_{\nu i}\bigg\}\frac{T^2}{12}\;, \qquad (5.80)
\end{aligned}
$$

where we inserted $d = 3$ in the last step, like already in Eq. (5.79).

Consider next the ghost loop; the vertex is in Eq. (5.50). Proceeding as in Eq. (5.70), we obtain

$$
\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,{\rm c}}
$$



$$
\begin{aligned}
&= -\frac{g^2}{2}\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q) \sum\!\!\!\!\!\!\!\int_{\tilde R_{\rm b},\tilde S_{\rm b},\tilde T_{\rm b}} \tilde{\bar c}^c(\tilde R)\tilde A^d_\alpha(\tilde S)\tilde c^e(\tilde T) \sum\!\!\!\!\!\!\!\int_{\tilde U_{\rm b},\tilde V_{\rm b},\tilde X_{\rm b}} \tilde{\bar c}^g(\tilde U)\tilde A^h_\beta(\tilde V)\tilde c^i(\tilde X)\Big\rangle_{0,{\rm c}} \times \\
&\qquad \times\, f^{cde}f^{ghi}\delta(-\tilde R + \tilde S + \tilde T)\delta(-\tilde U + \tilde V + \tilde X)\;\tilde R_\alpha \tilde U_\beta \\
&= g^2 \sum\!\!\!\!\!\!\!\int_{\tilde R_{\rm b},\tilde S_{\rm b},\tilde T_{\rm b},\tilde U_{\rm b},\tilde V_{\rm b},\tilde X_{\rm b}} \langle \tilde A^a_\mu(\tilde P)\tilde A^d_\alpha(\tilde S)\rangle_0\,\langle \tilde A^b_\nu(\tilde Q)\tilde A^h_\beta(\tilde V)\rangle_0\,\langle \tilde c^e(\tilde T)\tilde{\bar c}^g(\tilde U)\rangle_0\,\langle \tilde c^i(\tilde X)\tilde{\bar c}^c(\tilde R)\rangle_0 \times \\
&\qquad \times\, f^{cde}f^{ghi}\delta(-\tilde R + \tilde S + \tilde T)\delta(-\tilde U + \tilde V + \tilde X)\;\tilde R_\alpha \tilde U_\beta\;, \qquad (5.81)
\end{aligned}
$$

where the Grassmann nature of the ghosts induced a minus sign.

Inserting the gluon propagator from Eq. (5.63) and the ghost propagator from Eq. (5.46), let us inspect in turn colour indices, spacetime indices, and momenta. The colour contractions result in a factor

$$
\delta^{ad}\delta^{bh}\delta^{eg}\delta^{ic}f^{cde}f^{ghi} = f^{cae}f^{ebc} = -N_{\rm c}\delta^{ab}\;. \qquad (5.82)
$$

The spacetime indices can be transported to the momenta: $\delta_{\mu\alpha}\delta_{\nu\beta}\tilde R_\alpha \tilde U_\beta = \tilde R_\mu \tilde U_\nu$. The momentum dependence can be written as

$$
\begin{aligned}
&\frac{\delta(\tilde P + \tilde S)\delta(\tilde Q + \tilde V)\delta(\tilde T - \tilde U)\delta(\tilde X - \tilde R)}{\tilde P^2\tilde Q^2\tilde T^2\tilde X^2}\delta(-\tilde R + \tilde S + \tilde T)\delta(-\tilde U + \tilde V + \tilde X)\;\tilde R_\mu \tilde U_\nu \\
&= \frac{\delta(\tilde P + \tilde S)\delta(\tilde Q + \tilde V)\delta(\tilde T - \tilde U)\delta(\tilde X - \tilde R)}{\tilde P^2\tilde Q^2\tilde T^2\tilde X^2}\delta(-\tilde X - \tilde P + \tilde T)\delta(-\tilde T - \tilde Q + \tilde X)\;\tilde X_\mu \tilde T_\nu \\
&= \frac{\delta(\tilde P + \tilde S)\delta(\tilde Q + \tilde V)\delta(\tilde T - \tilde U)\delta(\tilde X - \tilde R)}{\tilde P^2\tilde Q^2\tilde T^2(\tilde T - \tilde P)^2}\delta(-\tilde X - \tilde P + \tilde T)\delta(\tilde P + \tilde Q)\;(\tilde T_\mu - \tilde P_\mu)\tilde T_\nu\;. \qquad (5.83)
\end{aligned}
$$

We can now integrate over $\tilde S, \tilde V, \tilde U, \tilde R$ and $\tilde X$. Thereby

$$
\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,{\rm c}} = -g^2 N_{\rm c}\frac{\delta^{ab}\delta(\tilde P + \tilde Q)}{(\tilde P^2)^2}\sum\!\!\!\!\!\!\!\int_{\tilde S_{\rm b}} \frac{1}{\tilde S^2(\tilde P - \tilde S)^2}(\tilde S_\mu - \tilde P_\mu)\tilde S_\nu\;, \qquad (5.84)
$$

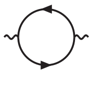
where we renamed $\tilde T \to \tilde S$.

The structure in Eq. (5.84) is identical to some of the terms in Eq. (5.74). In particular, if we again set the external momentum to zero, we return back to the integral in Eq. (5.76). Putting furthermore $d \to 3$, we arrive at

$$
\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,{\rm c}} \approx -g^2 N_{\rm c}\frac{\delta^{ab}\delta(\tilde P + \tilde Q)}{(\tilde P^2)^2}\bigg\{-\frac{1}{2}\,\delta_{\mu 0}\delta_{\nu 0} + \frac{1}{2}\,\delta_{\mu i}\delta_{\nu i}\bigg\}\frac{T^2}{12}\;. \qquad (5.85)
$$

Finally we consider the fermion loop; the vertex is in Eq. (5.51). Proceeding as in Eq. (5.70), we obtain

$$
\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Big(\frac{1}{2}\,S_I^2\Big)\Big\rangle_{0,{\rm c}}
$$



$$
\begin{aligned}
&= -\frac{g^2}{2}\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q) \sum\!\!\!\!\!\!\!\int_{\tilde R_\mathrm{f},\tilde S_\mathrm{b},\tilde T_\mathrm{f}} \tilde{\bar\psi}_A(\tilde R)\tilde\gamma_\alpha \tilde A^c_\alpha(\tilde S)\tilde\psi_B(\tilde T) \sum\!\!\!\!\!\!\!\int_{\tilde U_\mathrm{f},\tilde V_\mathrm{b},\tilde X_\mathrm{f}} \tilde{\bar\psi}_C(\tilde U)\tilde\gamma_\beta \tilde A^d_\beta(\tilde V)\tilde\psi_D(\tilde X)\Big\rangle_{0,\mathrm{c}} \times \\
&\quad \times \delta(-\tilde R+\tilde S+\tilde T)\delta(-\tilde U+\tilde V+\tilde X) T^c_{AB}T^d_{CD} \\
&= g^2 \sum\!\!\!\!\!\!\!\int_{\tilde R_\mathrm{f},\tilde S_\mathrm{b},\tilde T_\mathrm{f},\tilde U_\mathrm{f},\tilde V_\mathrm{b},\tilde X_\mathrm{f}} \langle \tilde A^a_\mu(\tilde P)\tilde A^c_\alpha(\tilde S)\rangle_0 \,\langle \tilde A^b_\nu(\tilde Q)\tilde A^d_\beta(\tilde V)\rangle_0 \,\mathrm{Tr}\Big[\langle\tilde\psi_D(\tilde X)\tilde{\bar\psi}_A(\tilde R)\rangle_0\,\tilde\gamma_\alpha\,\langle\tilde\psi_B(\tilde T)\tilde{\bar\psi}_C(\tilde U)\rangle_0\tilde\gamma_\beta\Big]\times \\
&\quad \times\, \delta(-\tilde R+\tilde S+\tilde T)\delta(-\tilde U+\tilde V+\tilde X) T^c_{AB}T^d_{CD}\;, \qquad (5.86)
\end{aligned}
$$

where the Grassmann nature of the fermions induced a minus sign. The capital indices include both colour and flavour.

Inserting the gluon propagator from Eq. (5.63) and the fermion propagator from Eq. (5.47), let us inspect in turn colour + flavour indices, the Lorentz indices, and momenta. The colour + flavour contractions result in a factor

$$
\delta^{ac}\delta^{bd}\delta_{DA}\delta_{BC}T^c_{AB}T^d_{CD} = \mathrm{Tr}\,[T^aT^b] = \frac{N_\mathrm{f}}{2}\;. \qquad (5.87)
$$

(For simplicity, we assumed that the flavours are degenerate in mass.) The spacetime indices yield

$$
\begin{aligned}
\delta_{\mu\alpha}\delta_{\nu\beta}\mathrm{Tr}\,[(-i\tilde{\not R}+m)\tilde\gamma_\alpha(-i\tilde{\not U}+m)\tilde\gamma_\beta] &= 4[-\tilde R_\sigma\tilde U_\rho(\delta_{\sigma\mu}\delta_{\rho\nu}-\delta_{\sigma\rho}\delta_{\mu\nu}+\delta_{\sigma\nu}\delta_{\rho\mu})+m^2\delta_{\mu\nu}] \\
&= 4[\delta_{\mu\nu}(\tilde R\cdot\tilde U+m^2)-\tilde R_\mu\tilde U_\nu-\tilde R_\nu\tilde U_\mu]\;. \qquad (5.88)
\end{aligned}
$$

The momentum dependence can be written as

$$
\begin{aligned}
&\frac{\delta(\tilde P+\tilde S)\delta(\tilde Q+\tilde V)\delta(\tilde X-\tilde R)\delta(\tilde T-\tilde U)}{\tilde P^2\tilde Q^2(\tilde X^2+m^2)(\tilde T^2+m^2)}\delta(-\tilde R+\tilde S+\tilde T)\delta(-\tilde U+\tilde V+\tilde X)\;f(\tilde R,\tilde U) \\
=\;&\frac{\delta(\tilde P+\tilde S)\delta(\tilde Q+\tilde V)\delta(\tilde X-\tilde R)\delta(\tilde T-\tilde U)}{\tilde P^2\tilde Q^2(\tilde X^2+m^2)(\tilde T^2+m^2)}\delta(-\tilde X-\tilde P+\tilde T)\delta(-\tilde T-\tilde Q+\tilde X)\;f(\tilde X,\tilde T) \\
=\;&\frac{\delta(\tilde P+\tilde S)\delta(\tilde Q+\tilde V)\delta(\tilde X-\tilde R)\delta(\tilde T-\tilde U)}{\tilde P^2\tilde Q^2[\tilde T^2+m^2][(\tilde T-\tilde P)^2+m^2]}\delta(-\tilde X-\tilde P+\tilde T)\delta(\tilde P+\tilde Q)\;f(\tilde T-\tilde P,\tilde T)\;. \qquad (5.89)
\end{aligned}
$$

We can now integrate over $\tilde S$, $\tilde V$, $\tilde R$, $\tilde U$ and $\tilde X$. Thereby

$$
\begin{aligned}
&\Big\langle \tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Big(\frac{1}{2}S_I^2\Big)\Big\rangle_{0,\mathrm{c}} \\
&\quad = 2g^2N_\mathrm{f}\frac{\delta^{ab}\delta(\tilde P+\tilde Q)}{(\tilde P^2)^2}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\delta_{\mu\nu}(\tilde S^2-\tilde P\cdot\tilde S+m^2)-2\tilde S_\mu\tilde S_\nu+\tilde P_\mu\tilde S_\nu+\tilde P_\nu\tilde S_\mu}{[\tilde S^2+m^2][(\tilde P-\tilde S)^2+m^2]}\;, \qquad (5.90)
\end{aligned}
$$

where we renamed $\tilde T\to\tilde S$.

The structure in Eq. (5.90) is again identical to some of the terms in Eq. (5.74), except that the Matsubara frequencies are fermionic. In particular, if we again set the external momentum to zero, and *also consider the limit $T\gg m$, so that quark masses can be ignored*, we get

$$
\begin{aligned}
\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\delta_{\mu\nu}\tilde S^2-2\tilde S_\mu\tilde S_\nu}{(\tilde S^2)^2} &= \delta_{\mu0}\delta_{\nu0}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\tilde S^2-2\tilde S_0^2}{(\tilde S^2)^2}+\delta_{\mu i}\delta_{\nu i}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\tilde S^2-2\tilde S_i^2}{(\tilde S^2)^2} \\
&= \delta_{\mu0}\delta_{\nu0}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\tilde S^2-2\tilde S_0^2}{(\tilde S^2)^2}+\delta_{\mu i}\delta_{\nu i}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\tilde S^2-(2/d)\mathbf{s}^2}{(\tilde S^2)^2} \\
&= \delta_{\mu0}\delta_{\nu0}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{\tilde S^2-2\tilde S_0^2}{(\tilde S^2)^2}+\delta_{\mu i}\delta_{\nu i}\sum\!\!\!\!\!\!\!\int_{\tilde S_\mathrm{f}}\frac{(1-2/d)\tilde S^2+(2/d)\tilde S_0^2}{(\tilde S^2)^2}\;. \qquad (5.91)
\end{aligned}
$$

The relation in Eq. (5.79) continues to hold in the fermionic case; setting $d \to 3$, we thereby get

$$\SumInt_{\tilde{S}_\mathrm{f}} \frac{\delta_{\mu\nu}\tilde{S}^2 - 2\tilde{S}_\mu\tilde{S}_\nu}{(\tilde{S}^2)^2} = \delta_{\mu 0}\delta_{\nu 0} \times 2I_T^\mathrm{f}(0) + \delta_{\mu i}\delta_{\nu i} \times \left(\frac{1}{3} - \frac{1}{3}\right) I_T^\mathrm{f}(0)\;. \tag{5.92}$$

The breaking of Lorentz invariance by the finite temperature is quite explicit here. Inserting finally $I_T^\mathrm{f}(0) = -T^2/24$ (cf. Eq. (4.74)), we arrive at

$$\left\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\left(\frac{1}{2}\,S_I^2\right)\right\rangle_{0,\mathrm{c}} \approx -g^2 N_\mathrm{f} \frac{\delta^{ab}\bar{\delta}(\tilde{P}+\tilde{Q})}{(\tilde{P}^2)^2}\left\{\delta_{\mu 0}\delta_{\nu 0} + 0 \times \delta_{\mu i}\delta_{\nu i}\right\}\frac{T^2}{6}\;. \tag{5.93}$$

Summing the contributions from Eqs. (5.68), (5.80), (5.85), (5.93), we get

$$\begin{aligned}
&\left\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\left(-S_I + \frac{1}{2}\,S_I^2\right)\right\rangle_{0,\mathrm{c}} \\
&\approx\; -g^2\frac{\delta^{ab}\bar{\delta}(\tilde{P}+\tilde{Q})}{(\tilde{P}^2)^2}\left\{\left[\left(3+\frac{3}{2}-\frac{1}{2}\right)N_\mathrm{c} + 2N_\mathrm{f}\right]\delta_{\mu 0}\delta_{\nu 0} + \left[\left(3-\frac{7}{2}+\frac{1}{2}\right)N_\mathrm{c}\right]\delta_{\mu i}\delta_{\nu i}\right\}\frac{T^2}{12} \\
&=\; -\frac{\delta^{ab}\bar{\delta}(\tilde{P}+\tilde{Q})}{(\tilde{P}^2)^2}\delta_{\mu 0}\delta_{\nu 0} \times g^2 T^2\left(\frac{N_\mathrm{c}}{3} + \frac{N_\mathrm{f}}{6}\right)\;.
\end{aligned} \tag{5.94}$$

It is very important to realise that all corrections have cancelled from the spatial part.

The result obtained has a direct physical meaning. Indeed, we recall from the discussion of scalar field theory, Eq. (3.78), that Eqs. (5.63), (5.94) can be interpreted as a (resummed) full propagator,

$$\left\langle \tilde{A}^a_\mu(\tilde{P})\tilde{A}^b_\nu(\tilde{Q})\right\rangle \approx \frac{\delta^{ab}\delta_{\mu\nu}\bar{\delta}(\tilde{P}+\tilde{Q})}{\tilde{P}^2 + \delta_{\mu 0}\delta_{\nu 0}\, m_\mathrm{E}^2}\;, \tag{5.95}$$

where

$$m_\mathrm{E}^2 \equiv g^2 T^2\left(\frac{N_\mathrm{c}}{3} + \frac{N_\mathrm{f}}{6}\right) \tag{5.96}$$

is called the *Debye mass parameter*. Its existence corresponds to the fact the colour-electric fields ($\equiv \tilde{A}_0$) get *screened* in a thermal plasma. In contrast, colour-magnetic fields ($\equiv A_i$) *do not get screened* — at least not at this order! We will return to the physical significance of these effects later on.