5.2. Gauge fixing and ghosts

Eq. (5.34) is gauge invariant and could be evaluated as such in lattice regularization, for instance. As before, we will here restrict to perturbation theory, however. Then gauge invariance needs once again to be broken, because the quadratic part of $\mathcal{L}_E$ does otherwise not contain an invertible matrix M, so that no propagators can be defined. For completeness, let us briefly recall the main steps of this procedure.

Let G^a be some function(al)s (no longer the Gauss law; the notation has changed!) of the path integration variables in Eq. (5.34), for instance $G^a(x) \equiv A_3^a(x)$ or $G^a(x) = -\partial_\mu A_\mu^a(x)$. The idea is to insert the object

$$\prod_{x,y,a,b} \delta(G^a) \det\Big[\frac{\delta G^a(x)}{\delta\theta^b(y)}\Big] \tag{5.35}$$

as a multiplier in front of the exponential in Eq. (5.34), in order to "remove the infinities from the integrations over the gauge orbits". Indeed, it is easy to see that this insertion does not change the value of gauge invariant expectation values (of course, in $\mathcal{Z}$ it induces an overall constant, analogous to C). First of all, since $\mathcal{L}_E$ is gauge invariant, its value does not depend (within each set of gauge-equivalent configurations) of the particular value selected by the constraint $G^a = 0$. Second, let us inspect the integration measure. We can imagine dividing the integration into one over gauge non-equivalent fields, $\bar{A}_\mu$, and gauge transformations thereof, parametrized by θ. Then

$$\begin{aligned}
&\int \mathcal{D}A_\mu\, \delta(G^a) \det\Big[\frac{\delta G^a}{\delta\theta^b}\Big] \exp\Big\{-\int_x \mathcal{L}_E(A_\mu)\Big\} = \\
&= \int \mathcal{D}\bar{A}_\mu \int \mathcal{D}\theta^b\, \delta(G^a) \det\Big[\frac{\delta G^a}{\delta\theta^b}\Big] \exp\Big\{-\int_x \mathcal{L}_E(\bar{A}_\mu)\Big\} \\
&= \int \mathcal{D}\bar{A}_\mu \int \mathcal{D}G^a\, \delta(G^a) \exp\Big\{-\int_x \mathcal{L}_E(\bar{A}_\mu)\Big\} \\
&= \int \mathcal{D}\bar{A}_\mu \exp\Big\{-\int_x \mathcal{L}_E(\bar{A}_\mu)\Big\}\,.
\end{aligned} \tag{5.36}$$

In other words, the dependence on the particular choice of the functions G^a disappears.

It is perhaps appropriate to point out that the somewhat formal manipulations in Eq. (5.36) can be given a precise meaning in lattice regularization, where the integration measure is defined as the gauge invariant Haar measure on SU(N_c). Nevertheless, the message remains the same as with our simplistic argument.

Given that the outcome is independent of G^a, it is conventional and convenient to furthermore replace $\delta(G^a)$ by $\delta(G^a - f^a)$, where f^a is some A_μ^a-independent function, and then to take an average over the f^a's with a Gaussian weight:

$$\begin{aligned}
\delta(G^a) \quad &\to \quad \int \mathcal{D}f^a\, \delta(G^a - f^a) \exp\Big(-\frac{1}{2\xi}\int_x f^a f^a\Big) \\
&= \quad \exp\Big(-\frac{1}{2\xi}\int_x G^a G^a\Big)\,.
\end{aligned} \tag{5.37}$$

Here an arbitrary parameter, ξ, has been inserted, in order to allow for a check later on that the results indeed are independent of its value.

Finally, the other structure in Eq. (5.35), the determinant, is conventionally written in terms of Faddeev-Popov ghosts, by making use of Eq. (4.46):

$$\det(M) = \int \mathcal{D}\bar{c}\mathcal{D}c\, \exp\Big(-\bar{c}Mc\Big)\,. \tag{5.38}$$

It should be noted, however, that since the "matrix" $\delta G^a/\delta\theta^b$ is purely bosonic, the ghost fields have to obey *the same boundary conditions as the gauge fields*, i.e. periodic, in spite of their Grassmann nature.

In total, then, we can write the gauge-fixed version of Eq. (5.34), adding now also Dirac fermions to complete the theory into QCD, as

$$\begin{aligned}\mathcal{Z}_{\rm phys} &= C\int_{\rm periodic}\mathcal{D}A^a_\mu\int_{\rm periodic}\mathcal{D}\bar{c}^a\,\mathcal{D}c^a\int_{\rm anti\text{-}periodic}\mathcal{D}\bar\psi\,\mathcal{D}\psi\,\times\\ &\times\ \exp\biggl\{-\int_0^\beta {\rm d}\tau\int_V {\rm d}^d\mathbf{x}\left[\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\frac{1}{2\xi}G^aG^a+\bar{c}^a\Bigl(\frac{\delta G^a}{\delta\theta^b}\Bigr)c^b+\bar\psi(\tilde\gamma_\mu\tilde D_\mu+m)\psi\right]\biggr\}\ ,\end{aligned}\tag{5.39}$$

where m is the diagonal quark mass matrix.

A particularly convenient gauge choice is that of *covariant gauges*. Then

$$G^a \equiv -\partial_\mu A^a_\mu\ ,\tag{5.40}$$

$$\frac{1}{2\xi}G^aG^a = \frac{1}{2\xi}\partial_\mu A^a_\mu\partial_\nu A^a_\nu\ ,\tag{5.41}$$

$$\frac{\delta G^a}{\delta\theta^b} = +\overleftarrow{\partial}_\mu\frac{\delta A^a_\mu}{\delta\theta^b} = \overleftarrow{\partial}_\mu\Bigl[\overrightarrow{\partial}_\mu\delta^{ab}+gf^{acb}A^c_\mu\Bigr]\ ,\tag{5.42}$$

$$\bar{c}^a\Bigl(\frac{\delta G^a}{\delta\theta^b}\Bigr)c^b = \partial_\mu\bar{c}^a\partial_\mu c^a+gf^{abc}\partial_\mu\bar{c}^aA^b_\mu c^c\ ,\tag{5.43}$$

where we made use of Eqs. (5.2), (5.6).

5.3. Feynman rules for Euclidean continuum QCD

For completeness, we now collect together the Feynman rules that apply to computations with the theory in Eq. (5.39), when the gauge is fixed according to Eq. (5.40).

Consider first the free (quadratic) part of the Euclidean action. Expressing everything in Fourier representation, this becomes

$$\begin{aligned}S_E &= \sum\!\!\!\!\!\!\!\int_{\tilde P_{\rm b},\tilde Q_{\rm b}}\bar\delta(\tilde P+\tilde Q)\biggl\{\frac{1}{2}\,i\tilde P_\mu\tilde A^a_\nu(\tilde P)\Bigl[i\tilde Q_\mu\tilde A^a_\nu(\tilde Q)-i\tilde Q_\nu\tilde A^a_\mu(\tilde Q)\Bigr]+\frac{1}{2\xi}i\tilde P_\mu\tilde A^a_\mu(\tilde P)i\tilde Q_\nu\tilde A^a_\nu(\tilde Q)\biggr\}+\\ &+\ \sum\!\!\!\!\!\!\!\int_{\tilde P_{\rm b},\tilde Q_{\rm b}}\bar\delta(-\tilde P+\tilde Q)\Bigl[-i\tilde P_\mu\tilde{\bar c}^a(\tilde P)i\tilde Q_\mu\tilde c^a(\tilde Q)\Bigr]+\sum\!\!\!\!\!\!\!\int_{\tilde P_{\rm f},\tilde Q_{\rm f}}\bar\delta(-\tilde P+\tilde Q)\tilde{\bar\psi}_A(\tilde P)[i\tilde\gamma_\mu\tilde Q_\mu+m]\tilde\psi_A(\tilde Q)\\ &= \sum\!\!\!\!\!\!\!\int_{\tilde P_{\rm b},\tilde Q_{\rm b}}\bar\delta(\tilde P+\tilde Q)\biggl\{\frac{1}{2}\,\tilde A^a_\mu(\tilde P)\tilde A^a_\nu(\tilde Q)\Bigl[\tilde P^2\delta_{\mu\nu}-\Bigl(1-\frac{1}{\xi}\Bigr)\tilde P_\mu\tilde P_\nu\Bigr]\biggr\}+\\ &+\ \sum\!\!\!\!\!\!\!\int_{\tilde P_{\rm b},\tilde Q_{\rm b}}\bar\delta(-\tilde P+\tilde Q)\Bigl[\tilde{\bar c}^a(\tilde P)\tilde c^a(\tilde Q)\tilde P^2\Bigr]+\sum\!\!\!\!\!\!\!\int_{\tilde P_{\rm f},\tilde Q_{\rm f}}\bar\delta(-\tilde P+\tilde Q)\tilde{\bar\psi}_A(\tilde P)[i\tilde{\not{P}}+m]\tilde\psi_A(\tilde Q)\ .\end{aligned}\tag{5.44}$$

For the quarks, the index A is assumed to comprise both colour and flavour indices, while in the Dirac space $\bar\psi$ and ψ are treated as vectors.

The propagators are obtained by inverting the matrices in Eq. (5.44):

$$\Bigl\langle\tilde A^a_\mu(\tilde P)\tilde A^b_\nu(\tilde Q)\Bigr\rangle_0 = \delta^{ab}\bar\delta(\tilde P+\tilde Q)\left[\frac{\delta_{\mu\nu}-\frac{\tilde P_\mu\tilde P_\nu}{\tilde P^2}}{\tilde P^2}+\frac{\xi\frac{\tilde P_\mu\tilde P_\nu}{\tilde P^2}}{\tilde P^2}\right]\ ,\tag{5.45}$$

$$\Bigl\langle\tilde c^a(\tilde P)\tilde{\bar c}^b(\tilde Q)\Bigr\rangle_0 = \delta^{ab}\bar\delta(\tilde P-\tilde Q)\frac{1}{\tilde P^2}\ ,\tag{5.46}$$

$$\left\langle \tilde{\psi}_A(\tilde{P})\tilde{\bar{\psi}}_B(\tilde{Q})\right\rangle_0 = \delta_{AB}\,\delta\!\!\!{}^-(\tilde{P}-\tilde{Q})\frac{-i\tilde{\not{P}}+m}{\tilde{P}^2+m^2}\,. \tag{5.47}$$

Finally, we list the interactions. It is convenient to symmetrize these through changes of integration and summation variables as much as possible. Thereby the three-gluon vertex becomes

$$\begin{aligned} S_I^{(AAA)} &= \int_x \frac{1}{2}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) g f^{abc} A_\mu^b A_\nu^c \\ &= \SumInt_{\tilde{P}_\mathrm{b},\tilde{Q}_\mathrm{b},\tilde{R}_\mathrm{b}} \frac{1}{3!}\tilde{A}_\mu^a(\tilde{P})\tilde{A}_\nu^b(\tilde{Q})\tilde{A}_\rho^c(\tilde{R})\,\delta\!\!\!{}^-(\tilde{P}+\tilde{Q}+\tilde{R})\ \times \\ &\qquad \times ig f^{abc}\Big[\delta_{\mu\rho}(\tilde{P}_\nu-\tilde{R}_\nu)+\delta_{\rho\nu}(\tilde{R}_\mu-\tilde{Q}_\mu)+\delta_{\nu\mu}(\tilde{Q}_\rho-\tilde{P}_\rho)\Big]\,. \end{aligned} \tag{5.48}$$

The four-gluon vertex reads

$$\begin{aligned} S_I^{(AAAA)} &= \int_x \frac{1}{4} g^4 f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e \\ &= \SumInt_{\tilde{P}_\mathrm{b},\tilde{Q}_\mathrm{b},\tilde{R}_\mathrm{b},\tilde{S}_\mathrm{b}} \frac{1}{4!}\tilde{A}_\mu^a(\tilde{P})\tilde{A}_\nu^b(\tilde{Q})\tilde{A}_\rho^c(\tilde{R})\tilde{A}_\sigma^d(\tilde{S})\,\delta\!\!\!{}^-(\tilde{P}+\tilde{Q}+\tilde{R}+\tilde{S})\times \\ &\times\ g^2\Big[f^{eab}f^{ecd}(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})+f^{eac}f^{ebd}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})+f^{ead}f^{ebc}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\rho}\delta_{\nu\sigma})\Big]\,. \end{aligned} \tag{5.49}$$

The ghost interaction can be written as

$$\begin{aligned} S_I^{(\bar{c}Ac)} &= \int_x \partial_\mu c^a g f^{abc} A_\mu^b c^c \\ &= \SumInt_{\tilde{P}_\mathrm{b},\tilde{Q}_\mathrm{b},\tilde{R}_\mathrm{b}} \tilde{\bar{c}}^a(\tilde{P})\tilde{A}_\mu^b(\tilde{Q})\tilde{c}^c(\tilde{R})\,\delta\!\!\!{}^-(-\tilde{P}+\tilde{Q}+\tilde{R})\Big(-igf^{abc}\tilde{P}_\mu\Big)\,. \end{aligned} \tag{5.50}$$

Finally, the fermion interaction is contained in

$$\begin{aligned} S_I^{(\bar{\psi}A\psi)} &= \int_x \bar{\psi}_A\tilde{\gamma}_\mu\Big(-igT_{AB}^a\Big)A_\mu^a\psi_B \\ &= \SumInt_{\tilde{P}_\mathrm{f},\tilde{Q}_\mathrm{b},\tilde{R}_\mathrm{f}} \tilde{\bar{\psi}}_A(\tilde{P})\tilde{\gamma}_\mu\tilde{A}_\mu^a(\tilde{Q})\tilde{\psi}_B(\tilde{R})\,\delta\!\!\!{}^-(-\tilde{P}+\tilde{Q}+\tilde{R})\Big(-igT_{AB}^a\Big)\,. \end{aligned} \tag{5.51}$$

5.4. Exercise 7

(a) Compute the free energy density $f(T)$ for free gluons (N_c colours) and massless quarks (N_f flavours), starting from Eq. (5.44).

(b) Deduce from here the result for usual electromagnetic blackbody radiation.

Note that ghosts play a role in both cases!

Solution to Exercise 7

(a) We remember from Eqs. (2.51), (2.80), (4.59), (4.72) that

$$J_b(0,T) = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}_b} \Big[\ln(\tilde{P}^2) - \text{const.}\Big] = -\frac{\pi^2 T^4}{90} , \tag{5.52}$$

$$J_f(0,T) = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}_f} \Big[\ln(\tilde{P}^2) - \text{const.}\Big] = \frac{7}{8}\frac{\pi^2 T^4}{90} . \tag{5.53}$$

We just need to figure out the prefactors of these terms. There will be contributions from gluons, ghosts and quarks, and we inspect them one at a time.

In the *gluonic* case, we are faced with the matrix

$$M_{\mu\nu} = \tilde{P}^2\delta_{\mu\nu} - \Big(1 - \frac{1}{\xi}\Big)\tilde{P}_\mu\tilde{P}_\nu . \tag{5.54}$$

Let us introduce two further matrices,

$$P^T_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{\tilde{P}_\mu\tilde{P}_\nu}{\tilde{P}^2} , \quad P^L_{\mu\nu} \equiv \frac{\tilde{P}_\mu\tilde{P}_\nu}{\tilde{P}^2} . \tag{5.55}$$

As matrices, these satisfy $P^T P^T = P^T$, $P^L P^L = P^L$, $P^T P^L = 0$, $P^T + P^L = \mathbb{1}$. Thereby P^T, P^L are projection operators, which implies that their eigenvalues are either zero or unity. The numbers of the unit eigenvalues can be found by taking the traces from the matrices: $\text{Tr}\,[P^T] = \delta_{\mu\mu} - 1 = d$, $\text{Tr}\,[P^L] = 1$.

We can now write

$$M_{\mu\nu} = \tilde{P}^2\, P^T_{\mu\nu} + \frac{1}{\xi}\tilde{P}^2\, P^L_{\mu\nu} . \tag{5.56}$$

Thereby M has d eigenvalues $\tilde{P}^2$ and one $\tilde{P}^2/\xi$. Also, there are $a = 1, \ldots, N_c^2 - 1$ copies of this structure. In total, then,

$$\begin{aligned} f(T)\big|_{\text{gluons}} &= (N_c^2 - 1)\bigg\{ d \times \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}_b} \Big[\ln(\tilde{P}^2) - \text{const.}\Big] + \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}_b} \Big[\ln(\frac{1}{\xi}\tilde{P}^2) - \text{const.}\Big]\bigg\} \\ &= (N_c^2 - 1)\bigg\{ -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}_b} \ln(\xi) + (d+1)J_b(0,T)\bigg\} . \end{aligned} \tag{5.57}$$

Furthermore, the first term in Eq. (5.57) vanishes in dimensional regularization, because it does not contain any scales.

For *ghosts*, the Gaussian integral yields (cf. Eq. (4.46))

$$\int \prod_a \mathrm{d}\bar{\tilde{c}}^a \mathrm{d}\tilde{c}^a \exp(-\bar{\tilde{c}}^a \tilde{P}^2 \tilde{c}^a) = \prod_a \tilde{P}^2 = \exp\Big\{ -\Big[-2(N_c^2 - 1)\frac{1}{2}\ln(\tilde{P}^2)\Big]\Big\} . \tag{5.58}$$

Furthermore we have to remember that ghosts obey periodic boundary conditions. Thereby

$$f(T)|_{\rm ghosts} = -2(N_{\rm c}^2-1)J_{\rm b}(0,T) \;. \tag{5.59}$$

Finally, *quarks* work out as in Eq. (4.51), except that they now come in $N_{\rm c}$ colours and $N_{\rm f}$ flavours:

$$f(T)|_{\rm quarks} = -4N_{\rm c}N_{\rm f}J_{\rm f}(0,T) \;. \tag{5.60}$$

Summing together Eqs. (5.57), (5.59), (5.60), inserting the values from Eqs. (5.52), (5.53), and setting $d=3$, we get

$$f(T)|_{\rm QCD} = -\frac{\pi^2T^4}{90}\Big[2(N_{\rm c}^2-1)+\frac{7}{2}\,N_{\rm f}N_{\rm c}\Big] \;. \tag{5.61}$$

This result is often referred to as the (QCD-version of the) Stefan-Boltzmann law.

It is important to realize (i) that the contribution from the ghosts is essential: according to Eq. (5.59), it cancels half of the result in Eq. (5.57), thereby yielding the correct number of physical degrees of freedom as a multiplier in Eq. (5.61); (ii) that the assumption that A_0^a has only periodic modes also played a role; had it also had antiperiodic ones, Eq. (5.61) would have obtained a further unphysical term (to be more precise, this statement assumes that the ghosts remain periodic; it might be possible to compensate for an antiperiodic part of A_0^a through an antiperiodic part in the ghost determinant, but the setup would then become rather complicated).

(b) The case of QED is obtained by setting $N_{\rm c}\to 1$, and $N_{\rm c}^2-1\to 1$ [since the group is U(1) rather than SU(1)]:

$$f(T)|_{\rm QED} = -\frac{\pi^2T^4}{90}\Big[2+\frac{7}{2}\,N_{\rm f}\Big] \;. \tag{5.62}$$

The factor 2 corresponds to the two photon polarizations; the factor 4, multiplying $\frac{7}{8}\,N_{\rm f}$, corresponds to a spin-$\frac{1}{2}$ particle and a spin-$\frac{1}{2}$ antiparticle. If neutrinos were included, they would only contribute with a factor $2\times\frac{7}{8}\,N_{\rm f}$.