5. Gauge fields

As with fermions in Sec. 4.2, our starting point with new kinds of fields is their classical Lagrangian in Minkowskian spacetime, which for non-Abelian gauge fields reads

$$\mathcal{L}_M = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \ , \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \ , \tag{5.1}$$

where g is the (bare) gauge coupling. Introducing a *covariant derivative in the adjoint representation*,

$$\mathcal{D}^{ac}_\mu \equiv \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu \ , \tag{5.2}$$

we note for later convenience that $F^a_{\mu\nu}$ can be expressed as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \mathcal{D}^{ac}_\nu A^c_\mu \quad (= \mathcal{D}^{ac}_\mu A^c_\nu - \partial_\nu A^a_\mu) \ . \tag{5.3}$$

We can naturally also supplement Eq. (5.1) with matter fields: for instance, letting ψ be a fermion in the fundamental representation, ϕ a scalar in the fundamental representation, and Φ a scalar in the adjoint representation, we could add

$$\delta\mathcal{L}_M = \bar\psi(i\gamma^\mu D_\mu - m)\psi + (D^\mu\phi)^\dagger D_\mu\phi + \mathcal{D}^{ac\mu}\Phi^c\mathcal{D}^{bd}_\mu\Phi^d - V(\phi^\dagger\phi, \Phi^a\Phi^a) \ , \tag{5.4}$$

where $D_\mu = \partial_\mu - igA^a_\mu T^a$ is a *covariant derivative in the fundamental representation*, and T^a are the Hermitean generators of $\mathrm{SU}(N_\mathrm{c})$, satisfying the algebra $[T^a, T^b] = i f^{abc} T^c$ and by convention normalised as $\mathrm{Tr}\,[T^a T^b] = \delta^{ab}/2$.

The construction principle behind Eqs. (5.1), (5.4) is *gauge invariance*. With $U \equiv \exp[ig\theta^a(x)T^a]$, the Lagrangian is invariant in the transformations $A_\mu \to A'_\mu$, $\psi \to \psi'$, $\phi \to \phi'$, $\Phi \to \Phi'$, with

$$A'_\mu \equiv A'^a_\mu T^a = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} = A_\mu + ig\theta^a[T^a, A_\mu] + T^a\partial^\mu\theta^a + \mathcal{O}(\theta^2) \tag{5.5}$$

$$\Leftrightarrow A'^a_\mu = A^a_\mu + \mathcal{D}^{ac}_\mu\theta^c + \mathcal{O}(\theta^2) \ , \tag{5.6}$$

$$\psi' = U\psi = (\mathbb{1} + ig\theta^a T^a)\psi + \mathcal{O}(\theta^2) \ , \tag{5.7}$$

$$\phi' = U\phi = (\mathbb{1} + ig\theta^a T^a)\phi + \mathcal{O}(\theta^2) \ , \tag{5.8}$$

$$\Phi' \equiv \Phi'^a T^a = U\Phi U^{-1} = \Phi + ig\theta^a[T^a, \Phi] + \mathcal{O}(\theta^2) \tag{5.9}$$

$$\Leftrightarrow \Phi'^a = \Phi^a + g f^{abc}\Phi^b\theta^c + \mathcal{O}(\theta^2) \ . \tag{5.10}$$

We would now like to quantise the theory in Eqs. (5.1), (5.4). At this point, the role of gauge invariance becomes rather convoluted, however. We may remember that:

- The classical theory is constructed by insisting on gauge invariance.
- Canonical quantization and the derivation of the Euclidean path integral necessitate an explicit breaking of gauge invariance.
- The final Euclidean path integral itself again displays gauge invariance.
- Yet carrying out perturbation theory with the Euclidean path integral necessitates once again an explicit breaking of gauge invariance.
- Nevertheless, only gauge invariant observables are considered to be physical.

So, conceptually, it is not a completely transparent setting!

In fact, as far as canonical quantization and the derivation of the Euclidean path integral are concerned, there are two procedures followed in the literature. The idea of the most common one is to carry out a complete gauge fixing (going to the axial gauge $A^a_3 = 0$), identifying physical degrees of freedom (A^a_1, A^a_2 and the corresponding canonical momenta; A^a_0 is expressed in terms of these by imposing a further constraint, the Gauss law); and then following the quantization procedure of scalar field theory.

We will here follow another approach, where the idea is to do *as little gauge fixing as possible*; the price to pay is that then one has to be careful about the *states* over which the physical Hilbert space is constructed[6]. The advantage of the approach is that the role of gauge invariance remains less compromised during quantization; if the evaluation of the resulting Euclidean path integral were also to be carried out non-perturbatively (within lattice regularization, for instance), then it would perhaps become clearer why only gauge invariant observables are physical.

5.1. Path integral for the partition function

For simplicity, let us restrict to Eq. (5.1) in the following. For canonical quantization, the first step is to construct the Hamiltonian. We will do this after setting

$$A^a_0 \equiv 0 \; , \tag{5.11}$$

which fixes the gauge only partially (according to Eq. (5.6), time-independent gauge transformations are still allowed, since A^a_0 remains zero in them). In some sense, the philosophy is to break gauge invariance only to the same "soft" degree that Lorentz invariance is also broken in the canonical formulation, through the special role that is given to the time coordinate.

The spatial components A^a_i are treated as the canonical coordinates. According to Eq. (5.3), $F^a_{0i} = \partial_0 A^a_i$, and Eq. (5.1) becomes

$$\mathcal{L}_M = \frac{1}{2}\, \partial_0 A^a_i \partial_0 A^a_i - \frac{1}{4}\, F^a_{ij} F^a_{ij} \; . \tag{5.12}$$

The canonical momenta corresponding to A^a_i, to be denoted by E^a_i, are

$$E^a_i \equiv \frac{\partial \mathcal{L}_M}{\partial(\partial_0 A^a_i)} = \partial_0 A^a_i \; , \tag{5.13}$$

and the Hamiltonian density reads

$$\mathcal{H} = E^a_i \partial_0 A^a_i - \mathcal{L}_M = \frac{1}{2}\, E^a_i E^a_i + \frac{1}{4}\, F^a_{ij} F^a_{ij} \; . \tag{5.14}$$

We also note that the "multiplier" of A^a_0 (before gauge fixing) reads, according to Eq. (5.3),

$$\frac{\delta S_M}{\delta A^a_0} = \frac{\delta}{\delta A^a_0} \int_x \left[\frac{1}{2}\, (\partial_0 A^b_i - \mathcal{D}^{bc}_i A^c_0) F^b_{0i} \right] = \mathcal{D}^{ab}_i F^b_{0i} \; , \tag{5.15}$$

where we made use of

$$\int_x f^a(x) \mathcal{D}^{ab}_\mu g^b(x) = - \int_x g^a(x) \mathcal{D}^{ab}_\mu f^b(x) \; . \tag{5.16}$$

The object in Eq. (5.15), the left-hand side of the Gauss law, will play an important role later on.

The theory is now canonically quantized by making A^a_i and E^a_i into operators, and by imposing the standard bosonic equal-time commutation relations between them,

$$[\hat{A}^a_i(t, \mathbf{x}), \hat{E}^b_j(t, \mathbf{y})] = i \delta^{ab} \delta_{ij} \delta(\mathbf{x} - \mathbf{y}) \; . \tag{5.17}$$

[6] This approach dates back to C.W. Bernard, *Feynman Rules for Gauge Theories at Finite Temperature,* Phys. Rev. D 9 (1974) 3312, and particularly D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and Instantons at Finite Temperature,* Rev. Mod. Phys. 53 (1981) 43.

The Hamiltonian becomes

$$\hat{H} = \int \mathrm{d}^d\mathbf{x} \biggl(\frac{1}{2}\, \hat{E}^a_i \hat{E}^a_i + \frac{1}{4}\, \hat{F}^a_{ij} \hat{F}^a_{ij} \biggr) \;. \tag{5.18}$$

A very important role in the quantization will be played by what we call the Gauss law operators. Combining Eq. (5.15) and $F^b_{0i} = \partial_0 A^b_i = E^b_i$, we write these as

$$\hat{G}^a = \hat{\mathcal{D}}^{ab}_i \hat{E}^b_i \;, \quad a = 1, \ldots, N_\mathrm{c}^2 - 1 \;. \tag{5.19}$$

Furthermore we define an operator parametrized by time-independent gauge transformations,

$$\hat{U} \equiv \exp\Bigl\{ -i \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) \hat{G}^a(\mathbf{x}) \Bigr\} \;. \tag{5.20}$$

We now claim that $\hat{U}$ *generates gauge transformations*. Let us prove this to leading non-trivial order in θ^a. First of all,

$$\begin{aligned}
\hat{U} \hat{A}^b_j(\mathbf{y}) \hat{U}^{-1} &= \hat{A}^b_j(\mathbf{y}) - i \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) [\hat{G}^a(\mathbf{x}), \hat{A}^b_j(\mathbf{y})] + \mathcal{O}(\theta^2) \\
&= \hat{A}^b_j(\mathbf{y}) - i \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) \Bigl\{ \partial^{\mathbf{x}}_i [\hat{E}^a_i(\mathbf{x}), \hat{A}^b_j(\mathbf{y})] + g f^{acd} \hat{A}^c_i(\mathbf{x}) [\hat{E}^d_i(\mathbf{x}), \hat{A}^b_j(\mathbf{y})] \Bigr\} + \mathcal{O}(\theta^2) \\
&= \hat{A}^b_j(\mathbf{y}) - \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) \Bigl\{ \partial^{\mathbf{x}}_i \delta^{ab} \delta_{ij} \delta(\mathbf{x}-\mathbf{y}) + g f^{acd} \hat{A}^c_i(\mathbf{x}) \delta^{db} \delta_{ij} \delta(\mathbf{x}-\mathbf{y}) \Bigr\} + \mathcal{O}(\theta^2) \\
&= \hat{A}^b_j(\mathbf{y}) + \partial_j \theta^b(\mathbf{y}) + g f^{bca} \hat{A}^c_i(\mathbf{y}) \theta^a(\mathbf{y}) + \mathcal{O}(\theta^2) \\
&= \hat{A}^b_j(\mathbf{y}) + \hat{\mathcal{D}}^{ba}_j \theta^a(\mathbf{y}) + \mathcal{O}(\theta^2) = \hat{A}'^b_j(\mathbf{y}) \;,
\end{aligned} \tag{5.21}$$

where in the last step we used Eq. (5.6). Similarly,

$$\begin{aligned}
\hat{U} \hat{E}^b_j(\mathbf{y}) \hat{U}^{-1} &= \hat{E}^b_j(\mathbf{y}) - i \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) [\hat{G}^a(\mathbf{x}), \hat{E}^b_j(\mathbf{y})] + \mathcal{O}(\theta^2) \\
&= \hat{E}^b_j(\mathbf{y}) - i \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) \Bigl\{ + g f^{acd} [\hat{A}^c_i(\mathbf{x}), \hat{E}^b_j(\mathbf{y})] \hat{E}^d_i(\mathbf{x}) \Bigr\} + \mathcal{O}(\theta^2) \\
&= \hat{E}^b_j(\mathbf{y}) + \int \mathrm{d}^d\mathbf{x}\, \theta^a(\mathbf{x}) g f^{acd} \delta^{cb} \delta_{ij} \delta(\mathbf{x}-\mathbf{y}) \hat{E}^d_i(\mathbf{x}) + \mathcal{O}(\theta^2) \\
&= \hat{E}^b_j(\mathbf{y}) + g f^{bda} \hat{E}^d_j(\mathbf{y}) \theta^a(\mathbf{y}) = \hat{E}'^b_j(\mathbf{y}) + \mathcal{O}(\theta^2) \;,
\end{aligned} \tag{5.22}$$

where we have identified the transformation law of an adjoint scalar according to Eq. (5.10).

One important consequence of Eqs. (5.21), (5.22) is that *the operators $\hat{G}^a$ commute with the Hamiltonian $\hat{H}$*. This follows from the fact that the Hamiltonian in Eq. (5.18) is gauge-invariant in time-independent gauge transformations (if $\hat{E}^a_i$ transforms as an adjoint scalar):

$$\hat{U} \hat{H} \hat{U}^{-1} = \hat{H} \Rightarrow [\hat{G}^a, \hat{H}] = 0 \;. \tag{5.23}$$

Another important consequence is that $\hat{U}$ transforms eigenstates as well: if $\hat{A}^a_i |A^a_i\rangle = A^a_i |A^a_i\rangle$, then

$$\hat{A}^a_i \hat{U}^{-1} |A^a_i\rangle = \hat{U}^{-1} \hat{A}'^a_i |A^a_i\rangle = \hat{U}^{-1} A'^a_i |A^a_i\rangle = A'^a_i \hat{U}^{-1} |A^a_i\rangle \;, \tag{5.24}$$

where we made use of Eq. (5.21). Consequently,

$$\hat{U}^{-1} |A^a_i\rangle = |A'^a_i\rangle \;. \tag{5.25}$$

Let us now define a physical state, $|\mathrm{phys}\rangle$, to be one which is gauge-invariant: $\hat{U}^{-1}|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$. Expanding to first order in θ^a, we see that physical states must satisfy

$$\hat{G}^a |\mathrm{phys}\rangle = 0 \;. \tag{5.26}$$

This is an operator manifestation of the statement that *physical states must obey the Gauss law.*

To summarise: since the Hamiltonian commutes with $\hat{G}^a$, we can choose the basis vectors of the Hilbert space to be simultaneous eigenstates of $\hat{H}$ and $\hat{G}^a$. Among all of these states, only the ones with zero eigenvalue of $\hat{G}^a$ are physical. It is then only these states which are to be used in the evaluation of $\mathcal{Z} = \mathrm{Tr}\,[\exp(-\beta\hat{H})]$, for instance.

After all these preparations, we are finally in a position to derive a path integral expression for $\mathcal{Z}$. In terms of basic quantum mechanics, we have a system with a Hamiltonian $\hat{H}$ and with a commuting quantity, $\hat{Q}$ (whose role is played by $\hat{G}^a$). One could consider the grand canonical partition function $\mathcal{Z}(T,\mu) = \mathrm{Tr}\,\{\exp[-\beta(\hat{H}-\mu\hat{Q})]\}$ but, according to what we have said, we are only interested in the contribution to $\mathcal{Z}$ from the states with zero "charge", $\hat{Q}|\mathrm{phys}\rangle = 0$. Assuming for concreteness that the eigenvalues q of $\hat{Q}$ are integers, we label the states with the eigenvalues E_q, q, so that $\hat{H}|E_q,q\rangle = E_q|E_q,q\rangle$, $\hat{Q}|E_q,q\rangle = q|E_q,q\rangle$. We can now write the relevant partition function by taking a trace over *all states*, but inserting a Kronecker delta function inside the trace:

$$\mathcal{Z}_{\mathrm{phys}} \equiv \sum_{E_0}\langle E_0,0|e^{\beta E_0}|E_0,0\rangle = \sum_{E_q,q}\langle E_q,q|\delta_{q,0}e^{\beta E_q}|E_q,q\rangle = \mathrm{Tr}\left[\delta_{\hat{Q},\hat{0}}e^{-\beta\hat{H}}\right] . \tag{5.27}$$

Since $\delta_{\hat{Q},\hat{0}} = \delta_{\hat{Q},\hat{0}}\delta_{\hat{Q},\hat{0}}$ and $[\hat{H},\hat{Q}] = 0$, we can finally write

$$\mathcal{Z}_{\mathrm{phys}} = \mathrm{Tr}\Big[\underbrace{\delta_{\hat{Q},\hat{0}}e^{-\epsilon\hat{H}}\delta_{\hat{Q},\hat{0}}e^{-\epsilon\hat{H}}\ldots\delta_{\hat{Q},\hat{0}}e^{-\epsilon\hat{H}}}_{N\text{ parts}}\Big] , \tag{5.28}$$

where $\epsilon = \beta/N$, $N\to\infty$ as before. We may represent

$$\delta_{\hat{Q},\hat{0}} = \int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_i}{2\pi}\,e^{i\theta_i\hat{Q}} = \int_{-\pi/\epsilon}^{\pi/\epsilon}\frac{\mathrm{d}c_i}{2\pi\epsilon^{-1}}\,e^{i\epsilon c_i\hat{Q}} , \tag{5.29}$$

and insert unity operators as in Eq. (1.34), but now the momentum state representation is placed between $\delta_{\hat{Q},\hat{0}}$ and $\exp(-\epsilon\hat{H})$. The typical building block then reads

$$\begin{aligned}&\langle x_{i+1}|e^{i\epsilon c_i\hat{Q}(\hat{x},\hat{p})}|p_i\rangle\langle p_i|e^{-\epsilon\hat{H}(\hat{p},\hat{x})}|x_i\rangle\\ &\quad= \exp\left\{-\epsilon\left[-ic_iQ(x_{i+1},p_i) + \frac{p_i^2}{2m} - ip_i\frac{x_{i+1}-x_i}{\epsilon} + V(x_i) + \mathcal{O}(\epsilon)\right]\right\} .\end{aligned} \tag{5.30}$$

It remains to: (i) take the limit $\epsilon\to 0$, whereby x_i, p_i, c_i become functions, $x(\tau), p(\tau), c(\tau)$; (ii) go from $d = 0$ to a general dimension; (iii) replace $x(\tau)\to A_i^a(x)$, $p(\tau)\to E_i^a(x)$, $c(\tau)\to\tilde{A}_0^a(x)$, $Q\to\mathcal{D}_i^{ab}E_i^b$, $m\to 1$. Then the integral over the square brackets in Eq. (5.30) becomes

$$\int_x\left[-i\tilde{A}_0^a\mathcal{D}_i^{ab}E_i^b + \frac{1}{2}E_i^aE_i^a - iE_i^a\partial_\tau A_i^a + \frac{1}{4}F_{ij}^aF_{ij}^a\right] = \int_x\left[\frac{1}{2}E_i^aE_i^a - iE_i^a\left(\partial_\tau A_i^a - \mathcal{D}_i^{ab}\tilde{A}_0^b\right) + \frac{1}{4}F_{ij}^aF_{ij}^a\right] , \tag{5.31}$$

where we made use of Eq. (5.16).

At this point we recognize inside the round brackets in Eq. (5.31) an expression of the form in Eq. (5.3). Of course, the field $\tilde{A}_0^a$ is *not* the original A_0^a-field, which was set to zero. Rather, it is a new field, which we are however free to *rename* to be A_0^a. Indeed, in the following we leave out the tilde from $\tilde{A}_0^a$, and redefine a *Euclidean* field strength tensor according to

$$F_{0i}^a \equiv \partial_\tau A_i^a - \mathcal{D}_i^{ab}\tilde{A}_0^b . \tag{5.32}$$

Noting furthermore that

$$\frac{1}{2}E_i^aE_i^a - iE_i^aF_{0i}^a = \frac{1}{2}(E_i^a - iF_{0i}^a)^2 + \frac{1}{2}F_{0i}^aF_{0i}^a , \tag{5.33}$$

we can carry out the Gaussian intergral over E_i^a, and end up with the desired path integral expression:

$$\mathcal{Z}_{\rm phys} = C \int \mathcal{D}A_0^a \int_{A_i^a(\beta,\mathbf{x})=A_i^a(0,\mathbf{x})} \mathcal{D}A_i^a \, \exp\left\{ - \int_0^\beta {\rm d}\tau \int_V {\rm d}^d\mathbf{x} \, \mathcal{L}_E \right\} , \quad \mathcal{L}_E = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \, . \tag{5.34}$$

Two remarks are in order on the final result in Eq. (5.34):

(i) The field A_0^a was introduce in order to impose the Gauss law at every value of τ; therefore, the integrations at each τ are independent of each other. In other words, it is not obvious from the derivation of the path integral whether the field A_0^a should satisfy periodic boundary conditions like the spatial components A_i^a do.

Now, any field defined on a compact interval $\tau \in (0,\beta)$ can be expressed as a sum of a periodic and an anti-periodic function: $A_0^a(\tau) = \frac{1}{2}\,[A_0^a(\tau) + A_0^a(\beta-\tau)] + \frac{1}{2}\,[A_0^a(\tau) - A_0^a(\beta-\tau)]$. The periodic part is what we would expect, and the question then is, what happens with the antiperiodic part? Note that this part could be Fourier-decomposed with fermionic Matsubara frequencies.

For the moment the answer does not seem obvious, but we will see in Exercise 7 that indeed only a periodic $A_0^a(\tau)$ leads to physical results.

(ii) For scalar field theory and fermions, Eqs. (2.7), (4.34), we found after a careful derivation of the Euclidean path integral that the result could be interpreted in terms of a simple recipe: $\mathcal{L}_E = -\mathcal{L}_M(t \to -i\tau)$. We may now ask whether the same is true for gauge fields as well?

A comparison of Eqs. (5.1), (5.34) shows that, indeed, the recipe again works; the only complication is that the Minkowskian A_0^a needs to be replaced with $i\tilde{A}_0^a$ (of which we have normally left out the tilde), just like ∂_t gets replaced with $i\partial_\tau$. This, of course, should be expected from gauge invariance, since covariant derivatives need to change as $D_t \to iD_\tau$.