According to Eq. (4.28), these satisfy the algebra

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu} \; , \quad \tilde{\gamma}_\mu^\dagger = \tilde{\gamma}_\mu \; . \tag{4.36}$$

We also define

$$\tilde{\partial}_0 \equiv \partial_\tau \; , \quad \tilde{\partial}_i \equiv \partial_i \; . \tag{4.37}$$

Thereby Eq. (4.34) can be written in the simple form

$$\mathcal{L}_E = \bar{\psi}[\tilde{\gamma}_\mu \tilde{\partial}_\mu + m]\psi \; , \tag{4.38}$$

and the partition function becomes

$$\mathcal{Z} = \int_{\substack{\bar{\psi}(\beta,\mathbf{x}) = -\bar{\psi}(0,\mathbf{x}) \\ \psi(\beta,\mathbf{x}) = -\psi(0,\mathbf{x})}} \mathcal{D}\bar{\psi}(\tau,\mathbf{x})\mathcal{D}\psi(\tau,\mathbf{x}) \, \exp\biggl\{ -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x}\, \mathcal{L}_E \biggr\} \; , \tag{4.39}$$

where we have substituted integration variables from $\psi^\dagger$ to $\bar{\psi}$, and set again $\hbar \to 1$. It is important to keep in mind that in the path integral formulation, ψ and $\bar{\psi}$ are independent integration variables.

In order to evaluate $\mathcal{Z}$, it is useful to go to Fourier space, like with the scalar fields. We write

$$\psi(x) \equiv \sumint_{\tilde{P}} e^{i\tilde{P}\cdot x} \tilde{\psi}(\tilde{P}) \; , \quad \bar{\psi}(x) \equiv \sumint_{\tilde{P}} e^{-i\tilde{P}\cdot x} \tilde{\bar{\psi}}(\tilde{P}) \; . \tag{4.40}$$

The anti-periodicity in Eq. (4.39) requires that $\tilde{P}$ be of the form

$$\tilde{P} = (\omega_n^{\mathrm{f}}, \mathbf{p}) \; , \quad e^{i\omega_n^{\mathrm{f}}\beta} = -1 \; , \tag{4.41}$$

whereby the *fermionic Matsubara frequencies* become

$$\omega_n^{\mathrm{f}} = 2\pi T\Bigl(n + \frac{1}{2}\Bigr) \; , \quad n \in \mathbb{Z} \; , \tag{4.42}$$

i.e. $\omega_n^{\mathrm{f}} = \pm\pi T, \pm 3\pi T, \ldots$. Note that anti-periodicity removes the Matsubara zero-mode from the spectrum. Correspondingly, recalling the discussion in Sec. 3.8, it is clear that there are *no infrared problems associated with fermions*!

In the Fourier representation, the exponent in Eq. (4.39) becomes

$$\begin{aligned} S_E &\equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x}\, \bar{\psi}(x)[\tilde{\gamma}_\mu\tilde{\partial}_\mu + m]\psi(x) \\ &= \int_x \sumint_{\tilde{P}_\mathrm{f}} \sumint_{\tilde{Q}_\mathrm{f}} e^{i(\tilde{P}-\tilde{Q})\cdot x} \tilde{\bar{\psi}}(\tilde{Q})[i\tilde{\gamma}_\mu\tilde{P}_\mu + m]\tilde{\psi}(\tilde{P}) \\ &= \sumint_{\tilde{P}_\mathrm{f}} \tilde{\bar{\psi}}(\tilde{P})[i\tilde{\not{P}} + m]\tilde{\psi}(\tilde{P}) \; , \end{aligned} \tag{4.43}$$

where we made use of Eq. (3.23), and defined $\tilde{\not{P}} \equiv \tilde{\gamma}_\mu \tilde{P}_\mu$. In contrast to real scalar fields, all Fourier modes are independent. Up to a constant, we can then also change the integration variables in Eq. (4.39) to be the Fourier modes. Furthermore, we note that

$$\int \mathrm{d}c^*\mathrm{d}c\, e^{-c^*ac} = \int \mathrm{d}c^*\mathrm{d}c\, [-c^*ac] = a \; , \tag{4.44}$$

$$\frac{\int \mathrm{d}c^*\mathrm{d}c\, c\, c^*\, e^{-c^*ac}}{\int \mathrm{d}c^*\mathrm{d}c\, e^{-c^*ac}} = \frac{\int \mathrm{d}c^*\mathrm{d}c\, c\, c^*}{\int \mathrm{d}c^*\mathrm{d}c\, [-c^*ac]} = \frac{1}{a} \; , \tag{4.45}$$

and recall that the generalizations to a multicomponent case read

$$\int \Bigl\{ \prod_i \mathrm{d}c_i^* \mathrm{d}c_i \Bigr\} \exp\Bigl(-c_i^* M_{ij} c_j \Bigr) = \det(M) \; , \tag{4.46}$$

$$\frac{\int \{ \prod_i \mathrm{d}c_i^* \mathrm{d}c_i \} \, c_k c_l^* \, \exp\Bigl(-c_i^* M_{ij} c_j \Bigr)}{\int \{ \prod_i \mathrm{d}c_i^* \mathrm{d}c_i \} \exp\Bigl(-c_i^* M_{ij} c_j \Bigr)} = (M^{-1})_{kl} \; . \tag{4.47}$$

Armoured with this knowledge, we can now work out the partition function $\mathcal{Z}$, as well as the propagator, which is needed for computing perturbative corrections to the partition function. From Eqs. (4.39), (4.43), (4.46),

$$\begin{aligned} \mathcal{Z} &= \tilde{C} \prod_{\tilde{P}_\mathrm{f}} \det[i\tilde{\not{P}} + m] \\ &= \tilde{C} \Bigl(\prod_{\tilde{P}_\mathrm{f}} \det[i\tilde{\not{P}} + m] \prod_{\tilde{P}_\mathrm{f}} \det[-i\tilde{\not{P}} + m] \Bigr)^{\frac{1}{2}} \; , \end{aligned} \tag{4.48}$$

where $\tilde{C}$ is some constant, and have we “replicated” the determinant and compensated for that by taking the square root. The reason for the replication is that

$$[i\tilde{\not{P}} + m][-i\tilde{\not{P}} + m] = \tilde{\not{P}}\,\tilde{\not{P}} + m^2 = (\tilde{P}^2 + m^2)\mathbb{1}_{4\times 4} \; , \tag{4.49}$$

where we made use of Eq. (4.36). Thereby

$$\mathcal{Z} = \tilde{C} \Bigl(\prod_{\tilde{P}_\mathrm{f}} \det[(\tilde{P}^2 + m^2)\mathbb{1}_{4\times 4}] \Bigr)^{\frac{1}{2}} = \tilde{C} \prod_{\tilde{P}_\mathrm{f}} (\tilde{P}^2 + m^2)^2 \; , \tag{4.50}$$

and the free energy density $f(T)$ becomes

$$\begin{aligned} f(T) &= \lim_{V\to\infty} \frac{F}{V} = \lim_{V\to\infty} \Bigl(-\frac{T}{V} \ln \mathcal{Z} \Bigr) \\ &= -\lim_{V\to\infty} \frac{T}{V} \times 2 \sum_{\tilde{P}_\mathrm{f}} \ln(\tilde{P}^2 + m^2) + \text{const.} \\ &= -4 \,\,{\displaystyle\sum\!\!\!\!\!\!\!\int_{\tilde{P}_\mathrm{f}}}\, \frac{1}{2} \ln(\tilde{P}^2 + m^2) + \text{const.} \; , \end{aligned} \tag{4.51}$$

where we identified the sum-integration measure from Eqs. (2.9), (2.55).

The following remarks are in order:

- the sum-integral appearing in Eq. (4.51) is similar to the bosonic one in Eq. (2.51), but is preceded by a minus-sign, and contains fermionic Matsubara frequencies. These are the characteristic properties of fermions.
- the factor 4 in Eq. (4.51) corresponds to the independent spin degrees of freedom contained in a Dirac spinor.
- like for the scalar field theory in Eq. (2.51) (or Eq. (2.24)), there is a constant part in the sum, independent of the energy (or mass). We will not specify it explicitly here; rather, there will be an implicit specification in Sec. 4.3, where we relate generic fermionic thermal sums to the already known bosonic ones.

Finally, from Eqs. (4.43), (4.47), we find the propagator:

$$\langle\tilde{\psi}_\alpha(\tilde{P})\,\tilde{\bar{\psi}}_\beta(\tilde{Q})\rangle_0 = \delta(\tilde{P}-\tilde{Q})[i\tilde{\not{P}}+m\mathbb{1}]^{-1}_{\alpha\beta} = \delta(\tilde{P}-\tilde{Q})\frac{[-i\tilde{\not{P}}+m\mathbb{1}]_{\alpha\beta}}{\tilde{P}^2+m^2} \,. \tag{4.52}$$

Once interactions are added, they can be reduced to a product of propagators with the Wick theorem in the same way as in Sec. 3.2 (remembering, though, that the Grassmann nature of the Dirac fields produces a minus-sign in every commutation). Finally, we reiterate that taking $m^2 \to 0$ in Eq. (4.52) does not lead to any divergences, because the denominator of a fermion propagator can never vanish at finite temperature (cf. Eq. (4.42)), unlike that of a boson.

4.3. Fermionic thermal sums

Let us now consider the same problem as in Sec. 2.3, but with fermionic Matsubara frequencies:

$$S_\mathrm{f} \equiv T \sum_{\omega_n^\mathrm{f}} f(\omega_n) \,. \tag{4.53}$$

For clarity we also denote the sum in Eq. (2.29) by S_b from now on. We can write:

$$\begin{aligned} S_\mathrm{f}(T) &= T[... + f(-3\pi T) + f(-\pi T) + f(\pi T) + ...] \\ &= T[... + f(-3\pi T) + f(-2\pi T) + f(-\pi T) + f(0) + f(\pi T) + f(2\pi T) + ...] \\ &\quad -T[... + f(-2\pi T) + f(0) + f(2\pi T) + ...] \\ &= 2\times\frac{T}{2}\Big[... + f\Big(-6\pi\frac{T}{2}\Big) + f\Big(-4\pi\frac{T}{2}\Big) + f\Big(-2\pi\frac{T}{2}\Big) + f(0) + f\Big(2\pi\frac{T}{2}\Big) + f\Big(4\pi\frac{T}{2}\Big) + ...\Big] \\ &\quad -T[... + f(-2\pi T) + f(0) + f(2\pi T) + ...] \\ &= 2S_\mathrm{b}\Big(\frac{T}{2}\Big) - S_\mathrm{b}(T) \,. \end{aligned} \tag{4.54}$$

Thereby all fermionic sums follow from the known bosonic ones!

To give an example, consider Eq. (2.34),

$$S_\mathrm{b}(T) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\,f(p) + \int_{-\infty-i0^+}^{+\infty-i0^+}\frac{\mathrm{d}p}{2\pi}\,[f(p)+f(-p)]n_\mathrm{B}(ip) \,. \tag{4.55}$$

Eq. (4.54) now implies

$$S_\mathrm{f}(T) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\,f(p) + \int_{-\infty-i0^+}^{+\infty-i0^+}\frac{\mathrm{d}p}{2\pi}\,[f(p)+f(-p)]\Big[2n_\mathrm{B}^{(T/2)}(ip) - n_\mathrm{B}^{(T)}(ip)\Big] \,. \tag{4.56}$$

Thereby the zero-temperature part remains unchanged, while the finite-temperature part has the new weight

$$\begin{aligned} 2n_\mathrm{B}^{(T/2)}(ip) - n_\mathrm{B}^{(T)}(ip) &= \frac{2}{\exp(2ip\beta)-1} - \frac{1}{\exp(ip\beta)-1} \\ &= \frac{1}{\exp(ip\beta)-1}\left[\frac{2}{\exp(ip\beta)+1} - 1\right] = \frac{1-\exp(ip\beta)}{[\exp(ip\beta)-1][\exp(ip\beta)+1]} \\ &= -n_\mathrm{F}^{(T)}(ip) \,, \end{aligned} \tag{4.57}$$

where $n_\mathrm{F}(p) \equiv 1/[\exp(\beta p)+1]$ is the Fermi-Dirac distribution function. In total, then, fermionic sums can be converted to integrals according to

$$S_\mathrm{f}(T) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\,f(p) - \int_{-\infty-i0^+}^{+\infty-i0^+}\frac{\mathrm{d}p}{2\pi}\,[f(p)+f(-p)]n_\mathrm{F}(ip) \,. \tag{4.58}$$

4.4. Exercise 6

(a) Defining

$$J_{\rm f}(m,T) = \frac{1}{2}\SumInt_{\tilde{P}_{\rm f}} \Big[\ln(\tilde{P}^2+m^2) - \text{const.}\Big] \;, \tag{4.59}$$

$$I_{\rm f}(m,T) = \SumInt_{\tilde{P}_{\rm f}} \frac{1}{\tilde{P}^2+m^2} \;, \tag{4.60}$$

and writing

$$J_{\rm f}(m,T) = J_0(m) + J_T^{\rm f}(m) \;, \quad I_{\rm f}(m,T) = I_0(m) + I_T^{\rm f}(m) \;, \tag{4.61}$$

find the general expressions for $J_T^{\rm f}(m)$, $I_T^{\rm f}(m)$.

(b) Derive the low-temperature and the high-temperature expansions for $J_T^{\rm f}(m)$, $I_T^{\rm f}(m)$. Note the absence of odd powers of m in the high-temperature expansions.

(c) Consider the fermionic version of Eq. (1.83),

$$G_{\rm f}(\tau) \equiv T \sum_{\omega_n^{\rm f}} \frac{e^{i\omega_n\tau}}{\omega_n^2+\omega^2} \;, \quad 0<\tau<\beta \;. \tag{4.62}$$

What is the explicit expression for $G_{\rm f}(\tau)$?

Solution to Exercise 6

(a) We proceed according to Eq. (4.54). From Eq. (2.50),

$$J_T^{\rm b}(m) = \int_{\mathbf{k}} T \ln\Big(1-e^{-\beta E_{\mathbf{k}}}\Big) \;, \tag{4.63}$$

so that

$$\begin{aligned} J_T^{\rm f}(m) &= \int_{\mathbf{k}} T\Big[\ln\Big(1-e^{-2\beta E_{\mathbf{k}}}\Big) - \ln\Big(1-e^{-\beta E_{\mathbf{k}}}\Big)\Big] \\ &= \int_{\mathbf{k}} T\ln\Big(1+e^{-\beta E_{\mathbf{k}}}\Big) \;. \end{aligned} \tag{4.64}$$

From Eq. (2.53),

$$I_T^{\rm b}(m) = \int_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} n_{\rm B}(E_{\mathbf{k}}) \;, \tag{4.65}$$

and then the same steps as in Eq. (4.57) lead to

$$I_T^{\rm f}(m) = -\int_{\mathbf{k}} \frac{1}{E_{\mathbf{k}}} n_{\rm F}(E_{\mathbf{k}}) \;. \tag{4.66}$$

(b) From Eq. (2.78), the low-temperature expansion for $J_T^{\rm b}$ reads

$$J_T^{\rm b}(m) \approx -T^4\bigg(\frac{m}{2\pi T}\bigg)^{\frac{3}{2}} e^{-\frac{m}{T}} \;. \tag{4.67}$$

In Eq. (4.54), the first term is exponentially suppressed, and thus

$$J_T^{\rm f}(m) \approx T^4\bigg(\frac{m}{2\pi T}\bigg)^{\frac{3}{2}} e^{-\frac{m}{T}} \;. \tag{4.68}$$

From Eq. (2.79), the low-temperature expansion for $I_T^{\rm b}$ reads

$$I_T^{\rm b}(m) \approx \frac{T^3}{m}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \;. \tag{4.69}$$

Again the first term in Eq. (4.54) is exponentially suppressed, so that

$$I_T^{\rm f}(m) \approx -\frac{T^3}{m}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \;. \tag{4.70}$$

From Eq. (2.81), the high-temperature expansion for $J_T^{\rm b}$ reads

$$J_T^{\rm b}(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{4\pi T}\right) - \frac{3}{4}\right] + \frac{m^6\zeta(3)}{3(4\pi)^4T^2} + \ldots \;. \tag{4.71}$$

According to Eq. (4.54), we then get

$$\begin{aligned}
J_T^{\rm f}(m) &= -\frac{1}{8}\frac{\pi^2T^4}{90} + \frac{1}{2}\frac{m^2T^2}{24} - \frac{m^3T}{12\pi} - 2\frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{4\pi T}\right) - \frac{3}{4} + \ln 2\right] + \frac{8m^6\zeta(3)}{3(4\pi)^4T^2} \\
&+ \frac{\pi^2T^4}{90} - \frac{m^2T^2}{24} + \frac{m^3T}{12\pi} + \frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{4\pi T}\right) - \frac{3}{4}\right] - \frac{m^6\zeta(3)}{3(4\pi)^4T^2} - \ldots \\
&= \frac{7}{8}\frac{\pi^2T^4}{90} - \frac{m^2T^2}{48} - \frac{m^4}{2(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{\pi T}\right) - \frac{3}{4}\right] + \frac{7m^6\zeta(3)}{3(4\pi)^4T^2} + \ldots \;.
\end{aligned} \tag{4.72}$$

We note that the cubic term indeed disappears from the difference. Finally, from Eq. (2.92), the high-temperature expansion for $I_T^{\rm b}$ reads

$$I_T^{\rm b}(m) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{2m^2}{(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{4\pi T}\right) - \frac{1}{2}\right] + \frac{2m^4\zeta(3)}{(4\pi)^4T^2} + \ldots \;, \tag{4.73}$$

and Eq. (4.54) then yields

$$\begin{aligned}
I_T^{\rm f}(m) &= \frac{1}{2}\frac{T^2}{12} - \frac{mT}{4\pi} - \frac{4m^2}{(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{4\pi T}\right) - \frac{1}{2} + \ln 2\right] + \frac{16m^4\zeta(3)}{(4\pi)^4T^2} \\
&- \frac{T^2}{12} + \frac{mT}{4\pi} + \frac{2m^2}{(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{4\pi T}\right) - \frac{1}{2}\right] - \frac{2m^4\zeta(3)}{(4\pi)^4T^2} + \ldots \\
&= -\frac{T^2}{24} - \frac{2m^2}{(4\pi)^2}\left[\ln\left(\frac{me^{\gamma_E}}{\pi T}\right) - \frac{1}{2}\right] + \frac{14m^4\zeta(3)}{(4\pi)^4T^2} + \ldots \;.
\end{aligned} \tag{4.74}$$

Again, the term odd in m has disappeared from the result.

(c) According to Eq. (1.88),

$$\begin{aligned}
G_{\rm b}(\tau) &= T\sum_{\omega_n^{\rm b}} \frac{e^{i\omega_n\tau}}{\omega_n^2+\omega^2} = \frac{1}{2\omega}\frac{\cosh\left[\left(\frac{\beta}{2}-\tau\right)\omega\right]}{\sinh\left[\frac{\beta\omega}{2}\right]} \\
&= \frac{1}{2\omega}\frac{e^{(\beta-\tau)\omega}+e^{\tau\omega}}{e^{\beta\omega}-1} = \frac{1}{2\omega}n_{\rm B}(\omega)\left[e^{(\beta-\tau)\omega}+e^{\tau\omega}\right] \;.
\end{aligned} \tag{4.75}$$

Employing Eq. (4.54), we get

$$\begin{aligned}
G_{\rm f}(\tau) &= \frac{1}{2\omega}\left\{\frac{2}{e^{2\beta\omega}-1}\left[e^{(2\beta-\tau)\omega}+e^{\tau\omega}\right] - \frac{1}{e^{\beta\omega}-1}\left[e^{(\beta-\tau)\omega}+e^{\tau\omega}\right]\right\} \\
&= \frac{1}{2\omega}\frac{1}{(e^{\beta\omega}-1)(e^{\beta\omega}+1)}\underbrace{\left\{2e^{(2\beta-\tau)\omega}+2e^{\tau\omega}-(e^{\beta\omega}+1)\left[e^{(\beta-\tau)\omega}+e^{\tau\omega}\right]\right\}}_{(e^{\beta\omega}-1)\left[e^{(\beta-\tau)\omega}-e^{\tau\omega}\right]} \\
&= \frac{1}{2\omega}n_{\rm F}(\omega)\left[e^{(\beta-\tau)\omega}-e^{\tau\omega}\right] \;.
\end{aligned} \tag{4.76}$$