4. Fermions

4.1. Path integral for the partition function of a fermionic harmonic oscillator

As in the bosonic case, the structure of the path integral can most easily be derived by considering a *non-interacting* field living in a *zero-dimensional* space ($d = 0$). We refer to this system as the harmonic oscillator.

In order to introduce the fermionic harmonic oscillator, let us start by briefly summarizing the main formulae for the bosonic case. In the operator description, the basic commutation relations, the Hamiltonian, the energy eigenstates, and the completeness relations, can be expressed as

$$[\hat{a}, \hat{a}] = 0 \,, \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \,, \quad [\hat{a}, \hat{a}^\dagger] = 1 \,; \tag{4.1}$$

$$\hat{H} = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big) = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \,; \tag{4.2}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \,, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \,, \quad n = 0, 1, 2, \ldots \,; \tag{4.3}$$

$$\mathbb{1} = \sum_n |n\rangle\langle n| = \int \mathrm{d}x\, |x\rangle\langle x| = \int \frac{\mathrm{d}p}{2\pi\hbar}\, |p\rangle\langle p| \,. \tag{4.4}$$

The observable we are interested in is $\mathcal{Z} = \mathrm{Tr}\,[\exp(-\beta\hat{H})]$, and the various path integral representations we obtained for this are (cf. Eqs. (1.37), (1.41), (1.44))

$$\mathcal{Z} = \int_{x(\beta\hbar)=x(0)} \frac{\mathcal{D}x\mathcal{D}p}{2\pi\hbar} \exp\bigg\{-\frac{1}{\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau \left[\frac{[p(\tau)]^2}{2m} - ip(\tau)\dot{x}(\tau) + V(x(\tau))\right]\bigg\} \tag{4.5}$$

$$= C \int_{x(\beta\hbar)=x(0)} \mathcal{D}x \exp\bigg\{-\frac{1}{\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau \left[\frac{m}{2}\Big(\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau}\Big)^2 + V(x(\tau))\right]\bigg\} \tag{4.6}$$

$$= C \int_{x(\beta\hbar)=x(0)} \mathcal{D}x \exp\Big(-\frac{1}{\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau\, \mathcal{L}_E\Big) \,, \quad \mathcal{L}_E = -\mathcal{L}_M(t \to -i\tau) \,, \tag{4.7}$$

where C is a constant.

In the fermionic case, we replace the algebra of Eq. (4.1) through

$$\{\hat{a}, \hat{a}\} = 0 \,, \quad \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0 \,, \quad \{\hat{a}, \hat{a}^\dagger\} = 1 \,. \tag{4.8}$$

Consider next the Hilbert space, the analogue of Eq. (4.3). We define a vacuum state $|0\rangle$ by

$$\hat{a}|0\rangle \equiv 0 \,. \tag{4.9}$$

Then we define a one-particle state $|1\rangle$ by

$$|1\rangle \equiv \hat{a}^\dagger|0\rangle \,. \tag{4.10}$$

It is now easy to see that the Hilbert space does not contain any other states: operating on $|1\rangle$ with $\hat{a}$ or $\hat{a}^\dagger$ gives either the already known state $|0\rangle$, or nothing:

$$\hat{a}|1\rangle = \hat{a}\hat{a}^\dagger|0\rangle = [1 - \hat{a}^\dagger\hat{a}]|0\rangle = |0\rangle \,, \tag{4.11}$$

$$\hat{a}^\dagger|1\rangle = \hat{a}^\dagger\hat{a}^\dagger|0\rangle = 0 \,. \tag{4.12}$$

The Hamiltonian, the analogue of Eq. (4.2), is an operator acting in this space, and can be defined as

$$\hat{H} \equiv \hbar\omega\Big(\hat{a}^\dagger\hat{a} - \frac{1}{2}\Big) = \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger) \,. \tag{4.13}$$

The observable remains the partition function,

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta\hat{H}}\right] = \langle 0|e^{-\beta\hat{H}}|0\rangle + \langle 1|e^{-\beta\hat{H}}|1\rangle \ . \tag{4.14}$$

Due to the simple structure of the Hilbert space, the partition function can be evaluated almost trivially:

$$\begin{aligned}\mathcal{Z} &= \left[\langle 0|0\rangle + \sum_{n=0}^{\infty}\frac{(-\beta\hbar\omega)^n}{n!}\underbrace{\langle 1|(\hat{a}^\dagger\hat{a})^n|1\rangle}_{1}\right]e^{\frac{\beta\hbar\omega}{2}} \\ &= \left[1+e^{-\beta\hbar\omega}\right]e^{\frac{\beta\hbar\omega}{2}} = 2\cosh\Bigl(\frac{\hbar\omega}{2T}\Bigr) \ , \end{aligned}\tag{4.15}$$

to be compared with Eq. (1.17). However, like in the bosonic case, it will ultimately be more useful to write a path integral representation for the partition function, rather than to remain with the operator formulation, and this will be our goal in the following.

As we recall from the bosonic case, the essential ingredient in the derivation of the path integral is a repeated use of the completeness relations in Eq. (4.4) (cf. Sec. 1.3). Therefore, we now need to find some analogues of Eqs. (4.4) for the fermionic system. This can be achieved with the help of *Grassmann variables*. In short, the answer will be that while in the bosonic case the system of Eqs. (4.1) leads to commuting classical fields, $x(\tau), p(\tau)$, $\tau = 0...\beta\hbar$, in the fermionic case the system of Eqs. (4.8) leads to anti-commuting Grassmann fields, $c(\tau), c^*(\tau)$, $\tau = 0...\beta\hbar$. Furthermore, while $x(\tau)$ is periodic (there is no periodicity constraint for $p(\tau)$), the fields $c(\tau), c^*(\tau)$ will both be anti-periodic over the compact τ-interval.

We define the Grassmann variables c, c^* (and, more generally, the Grassmann fields $c(\tau), c^*(\tau)$) through the following axioms:

- c, c^* are treated as independent variables, like x, p.
- $c^2 = (c^*)^2 \equiv 0$, $cc^* = -c^*c$.
- integration is defined through $\int \mathrm{d}c = \int \mathrm{d}c^* \equiv 0$, $\int \mathrm{d}c\, c = \int \mathrm{d}c^*\, c^* \equiv 1$.
- the integration is also Grassmann-like, in the sense that $\{c, \mathrm{d}c\} = \{c, \mathrm{d}c^*\} = \{c^*, \mathrm{d}c\} = \{c^*, \mathrm{d}c^*\} = 0$, $\{\mathrm{d}c, \mathrm{d}c\} = \{\mathrm{d}c, \mathrm{d}c^*\} = \{\mathrm{d}c^*, \mathrm{d}c^*\} = 0$.
- by convention we write the integration measure over both c and c^* in the order $\int \mathrm{d}c^*\mathrm{d}c$.
- a field $c(\tau)$ is a collection of independent Grassmann variables, one at each point $\tau \in (0, \beta\hbar)$.
- c, c^* are defined to anticommute with $\hat{a}, \hat{a}^\dagger$ as well, so that products like $c\hat{a}^\dagger$ act as regular bosonic operators, i.e. $[c\hat{a}^\dagger, c^*] = 0$.

We now define a ket-state, $|c\rangle$, and a bra-state, $\langle c|$, which are eigenstates of $\hat{a}$ ("from the left") and $\hat{a}^\dagger$ ("from the right"), respectively:

$$|c\rangle \equiv e^{-c\hat{a}^\dagger}|0\rangle = (1-c\hat{a}^\dagger)|0\rangle \ ; \quad \hat{a}|c\rangle = c|0\rangle = c|c\rangle \ , \tag{4.16}$$

$$\langle c| \equiv \langle 0|e^{-\hat{a}c^*} = \langle 0|(1-\hat{a}c^*) \ ; \quad \langle c|\hat{a}^\dagger = \langle 0|c^* = \langle c|c^* \ . \tag{4.17}$$

Such states possess the transition amplitude

$$\langle c'|c\rangle = \langle 0|(1-\hat{a}c'^*)(1-c\hat{a}^\dagger)|0\rangle = 1 + \langle 0|\hat{a}c'^*c\hat{a}^\dagger|0\rangle = 1 + c'^*c = e^{c'^*c} \ . \tag{4.18}$$

With these states, we note that we can define the objects we need:

$$\begin{aligned}
\int \mathrm{d}c^*\mathrm{d}c\, e^{-c^*c}|c\rangle\langle c| &= \int \mathrm{d}c^*\mathrm{d}c\,(1-c^*c)(1-c\hat{a}^\dagger)|0\rangle\langle 0|(1-\hat{a}c^*) \\
&= |0\rangle\langle 0| + \int \mathrm{d}c^*\mathrm{d}c\, c\,\hat{a}^\dagger|0\rangle\langle 0|\hat{a}\,c^* \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}\;, \qquad (4.19) \\
\int \mathrm{d}c^*\mathrm{d}c\, e^{-c^*c}\langle -c|\hat{A}|c\rangle &= \int \mathrm{d}c^*\mathrm{d}c\,(1-c^*c)\langle 0|(1+\hat{a}c^*)\hat{A}(1-c\hat{a}^\dagger)|0\rangle \\
&= \langle 0|\hat{A}|0\rangle - \int \mathrm{d}c^*\mathrm{d}c\,\langle 0|\hat{a}\,c^*\hat{A}\,c\,\hat{a}^\dagger|0\rangle \\
&= \langle 0|\hat{A}|0\rangle - \langle 1|\int \mathrm{d}c^*\mathrm{d}c\,c^*c\,\hat{A}|1\rangle \\
&= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle = \mathrm{Tr}\,[\hat{A}]\;, \qquad (4.20)
\end{aligned}$$

where we assumed $\hat{A}$ to be a "bosonic" operator, for instance the Hamiltonian.

Representing now the trace in Eq. (4.14) as in Eq. (4.20), and splitting up the exponential into a product of N small terms like in Eq. (1.31), we can write

$$\mathcal{Z} = \int \mathrm{d}c^*\mathrm{d}c\, e^{-c^*c}\langle -c|e^{-\frac{\epsilon\hat{H}}{\hbar}}\cdots e^{-\frac{\epsilon\hat{H}}{\hbar}}|c\rangle\;, \quad \epsilon \equiv \frac{\beta\hbar}{N}\;. \qquad (4.21)$$

Then we insert Eq. (4.19) in between the exponentials, as $\mathbb{1} = \int \mathrm{d}c_i^*\mathrm{d}c_i\, e^{-c_i^*c_i}|c_i\rangle\langle c_i|$. Thereby we are faced with objects like

$$\begin{aligned}
e^{-c_{i+1}^*c_{i+1}}\langle c_{i+1}|e^{-\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})}|c_i\rangle &\overset{(4.16),\,(4.17)}{=} \exp\Big(-c_{i+1}^*c_{i+1}\Big)\,\langle c_{i+1}|c_i\rangle\,\exp\Big[-\frac{\epsilon}{\hbar}H(c_{i+1}^*,c_i)\Big] \\
&\overset{(4.18)}{=} \exp\Big[-c_{i+1}^*c_{i+1} + c_{i+1}^*c_i - \frac{\epsilon}{\hbar}H(c_{i+1}^*,c_i)\Big] \\
&= \exp\bigg\{-\frac{\epsilon}{\hbar}\bigg[\hbar c_{i+1}^*\frac{c_{i+1}-c_i}{\epsilon} + H(c_{i+1}^*,c_i)\bigg]\bigg\}\;. \qquad (4.22)
\end{aligned}$$

Finally, some more attention needs to be paid to the right-most and left-most exponentials in Eq. (4.21). We may *define* $c_1 \equiv c$, which clarifies the fate of the right-most exponential, but the left-most needs to be inspected in detail:

$$\begin{aligned}
&\int \mathrm{d}c_1^*\mathrm{d}c_1\, e^{-c_1^*c_1}\langle -c_1|e^{-\frac{\epsilon}{\hbar}\hat{H}(\hat{a}^\dagger,\hat{a})}|\int \mathrm{d}c_N^*\mathrm{d}c_N|c_N\rangle \\
= &\int \mathrm{d}c_1^*\mathrm{d}c_1 \int \mathrm{d}c_N^*\mathrm{d}c_N\,\exp\Big[-c_1^*c_1 - c_1^*c_N - \frac{\epsilon}{\hbar}H(-c_1^*,c_N)\Big] \\
= &\int \mathrm{d}c_1^*\mathrm{d}c_1 \int \mathrm{d}c_N^*\mathrm{d}c_N\,\exp\bigg\{-\frac{\epsilon}{\hbar}\bigg[\hbar c_1^*\frac{c_1+c_N}{\epsilon} + H(-c_1^*,c_N)\bigg]\bigg\} \\
= &\int \mathrm{d}c_1^*\mathrm{d}c_1 \int \mathrm{d}c_N^*\mathrm{d}c_N\,\exp\bigg\{-\frac{\epsilon}{\hbar}\bigg[-\hbar c_1^*\frac{-c_1-c_N}{\epsilon} + H(-c_1^*,c_N)\bigg]\bigg\}\;. \qquad (4.23)
\end{aligned}$$

The total becomes

$$\begin{aligned}
\mathcal{Z} &= \int \mathrm{d}c_N^*\mathrm{d}c_N \cdots \int \mathrm{d}c_1^*\mathrm{d}c_1\,\exp\Big(-\frac{1}{\hbar}S_E\Big)\;, \qquad (4.24) \\
S_E &= \epsilon\sum_{i=1}^{N}\bigg[\hbar c_{i+1}^*\frac{c_{i+1}-c_i}{\epsilon} + H(c_{i+1}^*,c_i)\bigg]\bigg|_{c_{N+1}\equiv -c_1,\,c_{N+1}^*\equiv -c_1^*}\;. \qquad (4.25)
\end{aligned}$$

Finally, taking the formal limit $N \to \infty$, $\epsilon \to 0$, with $\beta\hbar = \epsilon N$ kept fixed, we arrive at

$$\mathcal{Z} = \int_{\substack{c^*(\beta\hbar)=-c^*(0)\\ c(\beta\hbar)=-c(0)}} \mathcal{D}c^*(\tau)\mathcal{D}c(\tau)\, \exp\biggl\{-\frac{1}{\hbar}\int_0^{\beta\hbar} \mathrm{d}\tau \left[\hbar c^*(\tau)\frac{\mathrm{d}c(\tau)}{\mathrm{d}\tau} + H(c^*(\tau), c(\tau))\right]\biggr\} . \tag{4.26}$$

To summarize, the fermionic path integral resembles the bosonic one in Eq. (4.5), but the Grassmann fields obey *antiperiodic boundary conditions*, unlike in the bosonic case.

In fact, the analogy between the bosonic and the fermionic path integrals can be pushed even further, but for that we need to specify precisely the form of the fermionic Hamiltonian, rather than just use the *ad hoc* definition in Eq. (4.13). To do this, we need to turn to Dirac fields.

4.2. The Dirac field at finite temperature

In order to make use of the result of the previous section, we need to construct the Hamiltonian for the Dirac field, and identify the objects that play the roles of the operators $\hat{a}, \hat{a}^\dagger$. Our starting point will be the "classical" Minkowskian Lagrangian,

$$\mathcal{L}_M = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi , \tag{4.27}$$

where

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 , \quad \{\gamma^\mu, \gamma^\nu\} \equiv 2\eta^{\mu\nu} , \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 , \quad m \equiv m\cdot \mathbb{1}_{4\times 4} . \tag{4.28}$$

The conjugate momentum is defined by

$$\pi = \frac{\partial\mathcal{L}_M}{\partial(\partial_0\psi)} = \bar{\psi}i\gamma^0 = i\psi^\dagger , \tag{4.29}$$

and the Hamiltonian density becomes

$$\mathcal{H} = \pi\partial_0\psi - \mathcal{L}_M = \bar{\psi}[-i\gamma^i\partial_i + m]\psi . \tag{4.30}$$

If we now go to the operator language and recall the canonical commutation relations,

$$\{\hat{\psi}_\alpha(x^0,\mathbf{x}), \hat{\psi}_\beta(x^0,\mathbf{y})\} = \{\hat{\psi}^\dagger_\alpha(x^0,\mathbf{x}), \hat{\psi}^\dagger_\beta(x^0,\mathbf{y})\} = 0 , \tag{4.31}$$

$$\{\hat{\psi}_\alpha(x^0,\mathbf{x}), \hat{\pi}_\beta(x^0,\mathbf{y})\} = \{\hat{\psi}_\alpha(x^0,\mathbf{x}), i\hat{\psi}^\dagger_\beta(x^0,\mathbf{y})\} = i\delta^{(d)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta} , \tag{4.32}$$

where the subscripts refer to the Dirac indices, $\alpha, \beta \in (1, ..., 4)$, we note that $\hat{\psi}_\alpha, \hat{\psi}^\dagger_\beta$ play precisely the same roles as $\hat{a}, \hat{a}^\dagger$ in Eq. (4.8), and that from the operator point of view the Hamiltonian has indeed the structure of Eq. (4.13),

$$\hat{H} = \int \mathrm{d}^d\mathbf{x}\, \hat{\psi}^\dagger(x^0,\mathbf{x})[-i\gamma^0\gamma^i\partial_i + m\gamma^0]\hat{\psi}(x^0,\mathbf{x}) . \tag{4.33}$$

Therefore, rephrasing Eq. (4.26) [we denote $c^* \to \psi^\dagger$, $c \to \psi$, and set again $\hbar = 1$], the object within square brackets in Eq. (4.26), which we *define* to be the Euclidean Lagrangian, reads now

$$\mathcal{L}_E \equiv \psi^\dagger\partial_\tau\psi + \psi^\dagger[-i\gamma^0\gamma^i\partial_i + m\gamma^0]\psi = \bar{\psi}[\gamma^0\partial_\tau - i\gamma^i\partial_i + m]\psi . \tag{4.34}$$

Most remarkably, a comparison of Eqs. (4.27) and (4.34) shows that our old recipe from Eq. (1.43), $\mathcal{L}_E = -\mathcal{L}_M(\tau = it)$, again works! (Note that $i\partial_0 = i\partial_t \to -\partial_\tau$.)

It is conventional and convenient to simplify the appearance of Eq. (4.34) by introducing so-called *Euclidean Dirac matrices* through

$$\tilde{\gamma}_0 \equiv \gamma^0 , \quad \tilde{\gamma}_i \equiv -i\gamma^i . \tag{4.35}$$