3.8. Proper free energy density to $\mathcal{O}(\lambda^{\frac{3}{2}})$: infrared resummation

We now consider the limit $m_{\rm phys} \to 0$, as would be the case (in perturbation theory) for, say, gluons in QCD. According to Eq. (3.80), this corresponds to $m_B \to 0$, since $I_0(0) = 0$ according to Eq. (2.73). Then we are faced with the infrared problem introduced in Sec. 3.5.

In the limit of a small mass, we can employ high-temperature expansions for the functions $J(m,T)$, $I(m,T)$, given in Eqs. (2.91), (2.94). Employing Eqs. (3.45), (3.55), we write the leading terms in the small-m_B expansion as

$$
\begin{aligned}
\mathcal{O}(\lambda_B^0): \quad f_{(0)}(T) &= J(m_B,T) = -\frac{\pi^2 T^4}{90} + \frac{m_B^2 T^2}{24} - \frac{m_B^3 T}{12\pi} + \mathcal{O}(m_B^4)\,, && (3.88)\\
\mathcal{O}(\lambda_B^1): \quad f_{(1)}(T) &= \frac{3}{4}\,\lambda_B [I(m_B,T)]^2 \\
&= \frac{3}{4}\,\lambda_B \left[\frac{T^2}{12} - \frac{m_B T}{4\pi} + \mathcal{O}(m_B^2)\right]^2 \\
&= \frac{3}{4}\,\lambda_B \left[\frac{T^4}{144} - \frac{m_B T^3}{24\pi} + \mathcal{O}(m_B^2 T^2)\right]\,, && (3.89)\\
\mathcal{O}(\lambda_B^2): \quad f_{(2)}(T) &= -\frac{9}{4}\,\lambda_B^2 \frac{T^4}{144}\frac{T}{8\pi m_B} + \mathcal{O}(m_B^0)\,. && (3.90)
\end{aligned}
$$

Let us inspect, in particular, *odd powers of* m_B. As is obvious from Eqs. (3.88)–(3.90), they are becoming increasingly important as we go further in the expansion. We remember from Sec. 2.6 that odd powers of m_B are necessarily associated with contributions from the Matsubara zero-modes. In fact, the odd power in Eq. (3.88) is directly the zero-mode contribution from Eq. (2.86),

$$
\delta_{\rm odd} f_{(0)} = J^{(n=0)} = -\frac{m_B^3 T}{12\pi}\,. \qquad (3.91)
$$

The odd power in Eq. (3.89) comes from a cross-term between the zero-mode contribution and the leading non-zero mode contribution:

$$
\delta_{\rm odd} f_{(1)} = \frac{3}{2}\,\lambda_B \times I'(0,T) \times I^{(n=0)} = -\frac{\lambda_B m_B T^3}{32\pi}\,. \qquad (3.92)
$$

Finally, the divergence in Eq. (3.90) comes from a product of two non-zero mode contributions with a particularly infrared sensitive zero-mode contribution:

$$
\delta_{\rm odd} f_{(2)} = \frac{9}{4}\,\lambda_B^2 \times [I'(0,T)]^2 \times \frac{{\rm d}I^{(n=0)}}{{\rm d}m_B^2} = -\frac{\lambda_B^2 T^5}{8^3 \pi m_B}\,. \qquad (3.93)
$$

Comparing these structures, we immediately see that the expansion parameter related to the odd powers is

$$
\frac{\delta_{\rm odd} f_{(1)}}{\delta_{\rm odd} f_{(0)}} \sim \frac{\delta_{\rm odd} f_{(2)}}{\delta_{\rm odd} f_{(1)}} \sim \frac{\lambda_B T^2}{8 m_B^2}\,. \qquad (3.94)
$$

Thus, if we try to take $m_B^2 \to 0$ (or even just $m_B^2 \ll \lambda_B T^2$), the loop expansion breaks down.

In order to cure this problem, our only hope is to *identify and sum the corresponding divergent terms to all orders*. We may then expect that the complete sum has a form where we can take $m_B^2 \to 0$, without meeting any more divergences. This procedure is often referred to as *resummation*.

Fortunately, it is indeed possible to identify the problematic terms. Eqs. (3.91)–(3.93) already show that at order N in λ_B, they are associated with terms containing N non-zero mode contributions $I'(0,T)$, and *one zero-mode contribution*. Graphically, this corresponds to a loop with one zero-mode line, dressed with N non-zero mode “bubbles”. Such graphs are usually called “ring diagrams” or, adopting botanistic terminology, “daisy” diagrams.

To be more quantitative, we consider Eq. (3.56) at order λ_B^N:

$$f(T) = \Big\langle S_I - \frac{1}{2} S_I^2 + \ldots + \frac{(-1)^{N+1}}{N!} S_I^N \Big\rangle_{0,\text{connected,drop overall} \int_x} \tag{3.95}$$

$$\Rightarrow \frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_B}{4}\right)^N \Big\langle \underbrace{\overset{6}{\phi\,\phi}\,\phi\,\phi \quad \phi\,\overset{6}{\phi\,\phi}\,\phi \quad \phi\,\overset{6}{\phi\,\phi}\,\phi}_{2(N-1)\quad 2(N-2)} \quad \cdots \quad \overset{6}{\phi\,\phi}\,\phi\,\phi \Big\rangle_{0,\ldots}$$

$$= \frac{(-1)^{N+1}}{N!}\left(\frac{\lambda_B}{4}\right)^N 6^N \underbrace{[2(N-1)][2(N-2)]...[2]}_{2^{N-1}(N-1)!} \underbrace{\left[\frac{T^2}{12}\right]^N}_{I'(0,T)} \underbrace{T \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}\left(\frac{1}{\mathbf{p}^2+m_B^2}\right)^N}_{\text{zero-mode ring}} .$$

Let us compute the zero-mode part (for simplicity we omit terms of $\mathcal{O}(\epsilon)$):

$$N=1: \quad \int \frac{\mathrm{d}^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{\mathbf{p}^2+m_B^2} = -\frac{m_B}{4\pi} = \frac{\mathrm{d}}{\mathrm{d}m_B^2}\left(-\frac{m_B^3}{6\pi}\right) ,$$

$$N=2: \quad \int \frac{\mathrm{d}^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{(\mathbf{p}^2+m_B^2)^2} = -\frac{\mathrm{d}}{\mathrm{d}m_B^2}\left(-\frac{m_B}{4\pi}\right) = -\frac{\mathrm{d}}{\mathrm{d}m_B^2}\frac{\mathrm{d}}{\mathrm{d}m_B^2}\left(-\frac{m_B^3}{6\pi}\right) ,$$

$$\begin{aligned}\text{generally}: \quad \int \frac{\mathrm{d}^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{(\mathbf{p}^2+m_B^2)^N} &= -\frac{1}{N-1}\frac{\mathrm{d}}{\mathrm{d}m_B^2}\int \frac{\mathrm{d}^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{(\mathbf{p}^2+m_B^2)^{N-1}} \\ &= \left(\frac{-1}{N-1}\right)\left(\frac{-1}{N-2}\right)\cdots\left(\frac{-1}{1}\right)\left(\frac{\mathrm{d}}{\mathrm{d}m_B^2}\right)^{N-1}\int \frac{\mathrm{d}^{3-2\epsilon}\mathbf{p}}{(2\pi)^{3-2\epsilon}} \frac{1}{\mathbf{p}^2+m_B^2} \\ &= \frac{(-1)^N}{(N-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}m_B^2}\right)^N \left(\frac{m_B^3}{6\pi}\right) .\end{aligned} \tag{3.96}$$

Combining Eqs. (3.95), (3.96), we get

$$\begin{aligned}\delta_{\text{odd}} f_{(N)} &= \frac{(-1)^{N+1}}{N!}\left(\frac{3\lambda_B}{2}\right)^N 2^{N-1}(N-1)!\left(\frac{T^2}{12}\right)^N T\frac{(-1)^N}{(N-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}m_B^2}\right)^N\left(\frac{m_B^3}{6\pi}\right) \\ &= -\frac{T}{2}\frac{1}{N!}\left(\frac{\lambda_B T^2}{4}\right)^N\left(\frac{\mathrm{d}}{\mathrm{d}m_B^2}\right)^N\left(\frac{m_B^3}{6\pi}\right) .\end{aligned} \tag{3.97}$$

As a crosscheck, it can easily be verified that this expression reproduces the explicit results for $N=0,1,2$ shown in Eqs. (3.91)–(3.93).

Now, thanks to the fact that Eq. (3.97) has precisely the right structure to correspond to a Taylor-expansion, we can sum the contributions in Eq. (3.97) to all orders:

$$\sum_{N=0}^{\infty}\frac{1}{N!}\left(\frac{\lambda_B T^2}{4}\right)^N\left(\frac{\mathrm{d}}{\mathrm{d}m_B^2}\right)^N\left(-\frac{m_B^3 T}{12\pi}\right) = -\frac{T}{12\pi}\left(m_B^2+\frac{\lambda_B T^2}{4}\right)^{\frac{3}{2}} . \tag{3.98}$$

A miracle has happened: from Eq. (3.98), the limit $m_B^2 \to 0$ can be taken without divergences! But there is a surprise: we get a contribution of $\mathcal{O}(\lambda_B^{3/2})$, rather than $\mathcal{O}(\lambda_B^2)$ as we naively expected in Sec. 3.5. In other words, infrared divergences in finite-temperature field theory modify even qualitatively the structure of the weak-coupling expansion.

Setting finally $m_B^2 \to 0$, and collecting all the finite terms from Eqs. (3.88)–(3.90), we find the correct expansion of $f(T)$ in the massless limit:

$$f(T) = -\frac{\pi^2 T^4}{90} - \frac{T}{12\pi}\left(\frac{\lambda_B T^2}{4}\right)^{3/2} + \frac{\lambda_B T^4}{4\times 48} + \mathcal{O}(\lambda_B^2 T^4) \tag{3.99}$$

$$= -\frac{\pi^2 T^4}{90}\left[1 - \frac{15}{32}\frac{\lambda_R}{\pi^2} + \frac{15}{16}\left(\frac{\lambda_R}{\pi^2}\right)^{\frac{3}{2}} + \mathcal{O}(\lambda_R^2)\right] . \tag{3.100}$$

Here we have inserted $\lambda_B = \lambda_R + \mathcal{O}(\lambda_R^2)$. Eq. (3.100) is our final result.

It is perhaps appropriate to add that despite the complications that we have found, higher order terms can also be added to Eq. (3.100). In fact, as of today (fall 2007), the coefficients of the five subsequent terms, of orders $\mathcal{O}(\lambda_R^2)$, $\mathcal{O}(\lambda_R^{5/2}\ln\lambda_R)$, $\mathcal{O}(\lambda_R^{5/2})$, $\mathcal{O}(\lambda_R^3\ln\lambda_R)$, and $\mathcal{O}(\lambda_R^3)$, are also known.[5] This progress is possible due to the fact that the resummation of higher order contributions that we have carried out explicitly in this section, can be implemented much more elegantly and systematically with so-called *effective field theory methods*. We return to the general procedure in later sections, but some flavour of the idea can perhaps be obtained by organizing the computation in yet another way, outlined in Exercise 5.

3.9. Exercise 5

(a) Following the zero-temperature computation of m^2_phys in Eq. (3.79), repeat the determination of the pole mass at $T \neq 0$, $m_B \to 0$. The result can be called the *thermal mass term*, m^2_eff.

(b) Argue that in the weak-coupling limit ($\lambda_R \ll 1$), the thermal mass term is important only for the Matsubara zero mode.

(c) Let us write the Lagrangian for $m_B^2 = 0$ as

$$\mathcal{L}_E = \underbrace{\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2_\text{eff}\phi^2_{n=0}}_{\mathcal{L}_0} + \underbrace{\frac{1}{4}\lambda_B\phi^4 - \frac{1}{2}m^2_\text{eff}\phi^2_{n=0}}_{\mathcal{L}_I} . \tag{3.101}$$

Treating $\mathcal{L}_0$ as the free theory, and $\mathcal{L}_I$ as an interaction of order λ_R, write down the contributions to $f_{(0)}$ and $f_{(1)}$. Check that once the computation is reorganized this way, the series is well-behaved, and the result agrees with what we got in Eq. (3.99).

Solution to Exercise 5

(a) The computation proceeds precisely like the one leading to Eq. (3.79), with the replacement $\int_{\tilde P} \to \SumInt_{\tilde P}$. Consequently,

$$m^2_\text{eff} = \lim_{m_B^2\to 0}\Big[m_B^2 + 3\lambda_B I(m_B,T)\Big] = 3\lambda_B I(0,T) = \frac{\lambda_R T^2}{4} + \mathcal{O}(\lambda_R^2) . \tag{3.102}$$

(b) For the non-zero Matsubara modes, with $\omega_n \neq 0$, we note that $\lambda_R T^2/4 \ll \omega_n^2$, so that the thermal mass term plays a subdominant role in the propagator. In contrast, for the Matsubara zero-mode, m^2_eff modifies the propagator significantly for $\mathbf{p}^2 \ll m^2_\text{eff}$, removing any infrared divergences.

(c) The free propagators are now different for the Matsubara zero-modes and non-zero modes:

$$\langle\tilde\phi'(\tilde P)\tilde\phi'(\tilde Q)\rangle_0 = \bar\delta(\tilde P + \tilde Q)\frac{1}{\omega_n^2 + \mathbf{p}^2} , \tag{3.103}$$

$$\langle\tilde\phi_{n=0}(\tilde P)\tilde\phi_{n=0}(\tilde Q)\rangle_0 = \bar\delta(\tilde P + \tilde Q)\frac{1}{\mathbf{p}^2 + m^2_\text{eff}} . \tag{3.104}$$

Eq. (3.16) gets replaced with

$$f_{(0)}(T) = \SumInt_{\tilde P}{}' \frac{1}{2}\ln(\tilde P^2) + T\int_{\mathbf{p}} \frac{1}{2}\ln(\mathbf{p}^2 + m^2_\text{eff}) - \text{const.}$$

[5] A. Gynther, M. Laine, Y. Schröder, C. Torrero and A. Vuorinen, *Four-loop pressure of massless O(N) scalar field theory,* JHEP 04 (2007) 094 [arXiv:hep-ph/0703307].

$$
\begin{aligned}
&= J'(0,T) + J^{(n=0)}(m_{\text{eff}},T) \\
&= -\frac{\pi^2 T^4}{90} - \frac{m_{\text{eff}}^3 T}{12\pi} \; .
\end{aligned} \tag{3.105}
$$

Eq. (3.17) now contains the two terms of the $\mathcal{L}_I$ in Eq. (3.101), and Eq. (3.18) becomes

$$
\begin{aligned}
f_{(1)}(T) &= \frac{3}{4}\,\lambda_B \langle \phi(0)\phi(0)\rangle_0 \langle \phi(0)\phi(0)\rangle_0 - \frac{1}{2}\, m_{\text{eff}}^2 \langle \phi_{n=0}(0)\phi_{n=0}(0)\rangle_0 \\
&= \frac{3}{4}\,\lambda_B \Big[I'(0,T) + I^{(n=0)}(m_{\text{eff}},T) \Big]^2 - \frac{1}{2}\, m_{\text{eff}}^2 I^{(n=0)}(m_{\text{eff}},T) \\
&= \frac{3}{4}\,\lambda_B \left[\frac{T^4}{144} - \frac{m_{\text{eff}} T^3}{24\pi} + \frac{m_{\text{eff}}^2 T^2}{16\pi^2} \right] + \frac{1}{2}\, m_{\text{eff}}^2 \frac{m_{\text{eff}} T}{4\pi} \; .
\end{aligned} \tag{3.106}
$$

Inserting Eq. (3.102) into the last term, we see that this contribution *precisely cancels* the "large" linear term within the square brackets. As we recall from Eq. (3.92), the linear term was part of the problematic series that needed to be resummed; hence the problematic series does not get generated at all with the present reorganization,

Combining Eqs. (3.105), (3.106), we then get

$$
f(T) = -\frac{\pi^2 T^4}{90} + \frac{3}{4}\,\lambda_R \frac{T^4}{144} - \frac{m_{\text{eff}}^3 T}{12\pi} + \mathcal{O}(\lambda_R^2) \; . \tag{3.107}
$$

which exactly agrees with Eq. (3.99).

The cancellation that took place in Eq. (3.106) can also be verified at higher orders. In particular, proceeding to $\mathcal{O}(\lambda_R^2)$, it can be seen that the structure in Eq. (3.93) gets cancelled as well: the resummation of infrared divergences that we carried out explicitly in Eq. (3.98) can indeed be fully captured by the reorganization in Eq. (3.101).