3.7. Proper free energy density to $\mathcal{O}(\lambda)$: ultraviolet renormalization

In Sec. 3.4 we attempted to compute the free energy density $f(T)$ of scalar field theory up to $\mathcal{O}(\lambda)$, but found a result which appeared divergent. Let us now show that, as must be the case in a renormalizable theory, the divergences disappear order-by-order in perturbation theory, if we *re-express* $f(T)$ *in terms of finite renormalized parameters.*

In order to proceed properly, we need to first change the notation somewhat. The zero-temperature parameters we employed before, m^2, λ, will now be re-interpreted as *bare parameters*, m_B^2, λ_B. (The temperature T, in contrast, is a physical property of the system, and is not subject to any modifications.) The expansion in Eq. (3.45) can then be written as

$$f(T) = \phi^{(0)}(m_B^2, T) + \lambda_B\, \phi^{(1)}(m_B^2, T) + \mathcal{O}(\lambda_B^2)\;. \tag{3.66}$$

As a second step, we introduce some *renormalized parameters*, m_R^2, λ_R. These could either be directly *physical quantities* (say, the mass of the scalar particle, and the scattering amplitude with some particular kinematics), or quantities which are not yet directly physical, but are related to physical quantities by finite equations (say, so-called $\overline{\text{MS}}$ scheme parameters). In any case, it is natural to choose the renormalized parameters such that in the limit of an extremely weak interaction, $\lambda_R \ll 1$, they formally agree with the bare parameters. In other words,

$$m_B^2 = m_R^2 + \lambda_R\, f^{(1)}(m_R^2) + \mathcal{O}(\lambda_R^2)\;, \tag{3.67}$$

$$\lambda_B = \lambda_R + \lambda_R^2\, g^{(1)}(m_R^2) + \mathcal{O}(\lambda_R^3)\;. \tag{3.68}$$

Note that renormalized parameters are defined at zero temperature, so no T can appear in these relations. The functions $f^{(i)}$ and $g^{(i)}$ are, in general, divergent in the limit that regularization is removed; for instance, in dimensional regularization, they contain poles like $1/\epsilon$.

The idea now is simply to convert the expansion in Eq. (3.66) into an expansion in λ_R, by inserting the expressions from Eqs. (3.67), (3.68), and Taylor-expanding in λ_R:

$$f(T) = \phi^{(0)}(m_R^2, T) + \lambda_R \biggl[\phi^{(1)}(m_R^2, T) + \frac{\partial \phi^{(0)}(m_R^2, T)}{\partial m_R^2} f^{(1)}(m_R^2) \biggr] + \mathcal{O}(\lambda_R^2)\;. \tag{3.69}$$

We note that to $\mathcal{O}(\lambda_R^2)$, only the mass parameter needs to be renormalized.

To carry out the renormalization in practice, we need to choose a *scheme*. We will here choose the so-called *pole mass scheme*, where m_R^2 is taken to be the physical mass squared of the ϕ-particle, denoted by m_phys^2. In Minkowskian spacetime, this appears as the exponential time-evolution factor,

$$e^{-iE_0 t} \equiv e^{-i m_\text{phys} t}\;, \tag{3.70}$$

in the propagator of a particle at rest, $\mathbf{p} = \mathbf{0}$. In Euclidean spacetime, this corresponds to the exponential fall-off, $\exp(-m_\text{phys}\tau)$, of the propagator. Therefore, in order to determine m_phys^2 to $\mathcal{O}(\lambda_R)$, we need to compute the *full propagator*, $G(x)$, to $\mathcal{O}(\lambda_R)$ at zero temperature.

The full propagator can be defined as the generalization of Eq. (3.37) to the interacting case:

$$\begin{aligned} G(x) &\equiv \frac{\langle \phi(x)\phi(0)\exp(-S_I)\rangle_0}{\langle \exp(-S_I)\rangle_0} \\ &= \frac{\langle \phi(x)\phi(0)\rangle_0 - \langle \phi(x)\phi(0) S_I\rangle_0 + \mathcal{O}(\lambda_B^2)}{1 - \langle S_I\rangle_0 + \mathcal{O}(\lambda_B^2)} \\ &= \langle \phi(x)\phi(0)\rangle_0 - \Bigl[\langle \phi(x)\phi(0) S_I\rangle_0 - \langle \phi(x)\phi(0)\rangle_0 \langle S_I\rangle_0 \Bigr] + \mathcal{O}(\lambda_B^2)\;. \end{aligned} \tag{3.71}$$

We may recall from Quantum Field Theory that the second term inside the square brackets serves to cancel disconnected contractions, just like the subtractions in Eq. (3.10) did for the free energy density. Therefore, we will drop the second term in the following, and replace the expectation value in the first term by $\langle ... \rangle_{0,\mathrm{c}}$, like we already did in Eq. (3.21).

Now, let us start by inspecting the leading (zeroth order) term in Eq. (3.71), in order learn how m_{phys} can be conveniently extracted from the propagator. Introducing the notation

$$\int_{\tilde{P}} \equiv \lim_{T\to 0} \sumint_{\tilde{P}} = \int \frac{\mathrm{d}^{d+1}\tilde{P}}{(2\pi)^{d+1}} , \tag{3.72}$$

the free propagator reads (cf. Eq. (3.26))

$$G_0(x) = \langle \phi(x)\phi(0)\rangle_0 = \int_{\tilde{P}} \frac{e^{i\tilde{P}\cdot x}}{\tilde{P}^2 + m^2} . \tag{3.73}$$

For Eq. (3.70), we need to project to zero spatial momentum, $\mathbf{p} = \mathbf{0}$; evidently, this can be achieved by taking a spatial average of $G_0(x)$:

$$\int \mathrm{d}^d\mathbf{x}\, \langle \phi(\tau,\mathbf{x})\phi(0)\rangle_0 = \int \frac{\mathrm{d}p_0}{2\pi} \frac{e^{ip_0\tau}}{p_0^2 + m^2} . \tag{3.74}$$

We see that we get an integral which can be evaluated with the help of the Cauchy theorem and, in particular, that the exponential fall-off of the correlation function is determined by the pole position of the momentum-space propagator:

$$\int \mathrm{d}^d\mathbf{x}\, \langle \phi(\tau,\mathbf{x})\phi(0)\rangle_0 = \frac{1}{2\pi} 2\pi i \frac{e^{-m\tau}}{2im} , \quad \tau \geq 0 . \tag{3.75}$$

Hence,

$$m^2_{\mathrm{phys}}\big|_{\lambda=0} = m^2 , \tag{3.76}$$

and, more generally, *the physical mass can be extracted by determining the pole position of the full propagator in momentum space.*

We then proceed to the second term in Eq. (3.71):

$$\begin{aligned}
-\langle \phi(x)\phi(0)S_I\rangle_{0,\mathrm{c}} &= -\frac{\lambda_B}{4}\int_z \langle \phi(x)\phi(0)\ \phi(z)\phi(z)\phi(z)\phi(z)\rangle_{0,\mathrm{c}} \\
&= -\frac{\lambda_B}{4}\int_z 4\times 3\, \langle \phi(x)\phi(z)\rangle_0\ \langle \phi(z)\phi(0)\rangle_0\ \langle \phi(z)\phi(z)\rangle_0 \\
&= -3\lambda_B G_0(0) \int_z G_0(z)G_0(x-z) \\
&= -3\lambda_B \int_{\tilde{P}} \frac{1}{\tilde{P}^2 + m_B^2} \int_z \int_{\tilde{Q},\tilde{R}} e^{i\tilde{Q}\cdot z} e^{i\tilde{R}\cdot(x-z)} \frac{1}{\tilde{Q}^2 + m_B^2}\frac{1}{\tilde{R}^2 + m_B^2} \\
&= -3\lambda_B I_0(m_B) \int_{\tilde{R}} \frac{e^{i\tilde{R}\cdot x}}{(\tilde{R}^2 + m_B^2)^2} .
\end{aligned} \tag{3.77}$$

Summing together with Eq. (3.73), the full propagator reads

$$\begin{aligned}
G(x) &= \int_{\tilde{P}} e^{i\tilde{P}\cdot x}\left[\frac{1}{\tilde{P}^2 + m_B^2} - 3\lambda_B I_0(m_B)\frac{1}{(\tilde{P}^2 + m_B^2)^2} + \mathcal{O}(\lambda_B^2)\right] \\
&= \int_{\tilde{P}} \frac{e^{i\tilde{P}\cdot x}}{\tilde{P}^2 + m_B^2 + 3\lambda_B I_0(m_B)} + \mathcal{O}(\lambda_B^2) ,
\end{aligned} \tag{3.78}$$

where we have effectively resummed a series of higher order corrections.

The same steps that lead us from Eq. (3.74) to (3.76) now produce

$$m^2_{\mathrm{phys}} = m_B^2 + 3\lambda_B I_0(m_B) + \mathcal{O}(\lambda_B^2) . \tag{3.79}$$

Recalling from Eq. (3.68) that $m_B^2 = m_R^2 + \mathcal{O}(\lambda_R)$, $\lambda_B = \lambda_R + \mathcal{O}(\lambda_R^2)$, this relation can be inverted, to give

$$m_B^2 = m_{\rm phys}^2 - 3\lambda_R I_0(m_{\rm phys}) + \mathcal{O}(\lambda_R^2) \; . \tag{3.80}$$

This corresponds precisely to Eq. (3.67). The function I_0, given in Eq. (2.73), diverges in the limit $\epsilon \to 0$,

$$I_0(m_{\rm phys}) = -\frac{m_{\rm phys}^2}{16\pi^2}\mu^{-2\epsilon}\left[\frac{1}{\epsilon} + \ln\frac{\bar\mu^2}{m_{\rm phys}^2} + 1 + \mathcal{O}(\epsilon)\right] \; , \tag{3.81}$$

and we may hope that the divergence cancels the unphysical ones that we found in $f(T)$.

Indeed, let us take the step from Eq. (3.66) to Eq. (3.69), employing the explicit expression from Eq. (3.45),

$$f(T) = J(m_B, T) + \frac{3}{4}\,\lambda_B[I(m_B,T)]^2 + \mathcal{O}(\lambda_B^2) \; . \tag{3.82}$$

Recalling from Eq. (2.52) that

$$I(m,T) = \frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}m}J(m,T) = 2\frac{\mathrm{d}}{\mathrm{d}m^2}J(m,T) \; , \tag{3.83}$$

we can expand the two terms in Eq. (3.82) as a Taylor series around $m_{\rm phys}^2$,

$$\begin{aligned} J(m_B,T) &= J(m_{\rm phys},T) + (m_B^2 - m_{\rm phys}^2)\frac{\partial J(m_{\rm phys},T)}{\partial m_{\rm phys}^2} + \mathcal{O}(\lambda_R^2) \\ &= J(m_{\rm phys},T) - \frac{3}{2}\,\lambda_R I_0(m_{\rm phys}) I(m_{\rm phys},T) + \mathcal{O}(\lambda_R^2) \; , \end{aligned} \tag{3.84}$$

$$\lambda_B[I(m_B,T)]^2 = \lambda_R[I(m_{\rm phys},T)]^2 + \mathcal{O}(\lambda_R^2) \; , \tag{3.85}$$

where we inserted Eq. (3.80). Eq. (3.82) then becomes

$$\begin{aligned} f(T) &= J(m_{\rm phys},T) + \frac{3}{4}\,\lambda_R\Big[I^2(m_{\rm phys},T) - 2I_0(m_{\rm phys})\,I(m_{\rm phys},T)\Big] + \mathcal{O}(\lambda_R^2) \\ &= \underbrace{\left\{J_0(m_{\rm phys}) - \frac{3}{4}\,\lambda_R I_0^2(m_{\rm phys})\right\}}_{T=0\text{ part}} + \underbrace{\left\{J_T(m_{\rm phys}) + \frac{3}{4}\,\lambda_R I_T^2(m_{\rm phys})\right\}}_{T\neq 0\text{ part}} + \mathcal{O}(\lambda_R^2) \; , \end{aligned} \tag{3.86}$$

where we have inserted the definitions from Eq. (2.56), $J(m,T) = J_0(m) + J_T(m)$, $I(m,T) = I_0(m) + I_T(m)$.

Recalling Eqs. (2.72), (2.73), the first term in Eq. (3.86), the "$T = 0$ part", is divergent. However, this term is completely independent of the temperature. Thus it does not play a role in thermodynamics, but rather corresponds to a *vacuum energy density*: it only plays a physical role in connection with gravity. If we included gravity, however, we should also include a bare cosmological constant, Λ_B, to the bare Lagrangian; it would contribute additively to Eq. (3.86), and we can simply define

$$\Lambda_{\rm phys} \equiv \Lambda_B + J_0(m_{\rm phys}) - \frac{3}{4}\,\lambda_R I_0^2(m_{\rm phys}) + \mathcal{O}(\lambda_R^2) \; . \tag{3.87}$$

The divergences are now "eaten up" by Λ_B, and $\Lambda_{\rm phys}$ is finite.

In contrast, the second term in Eq. (3.86), the "$T \neq 0$ part", is finite: it contains the functions J_T, I_T for which we have determined analytically various limiting values in Eqs. (2.78), (2.79), (2.81), (2.92), and general integral expressions in Eqs. (2.75), (2.76). Therefore all thermodynamic quantities obtained from derivatives of $f(T)$, such as the entropy density, specific heat, etc, are manifestly finite. In other words, the temperature-dependent ultraviolet divergences that we found in Sec. 3.4 have indeed disappeared through zero-temperature renormalization, as must be the case.