3.4. Naive free energy density to $\mathcal{O}(\lambda)$: ultraviolet divergences

We now return to the free energy density of scalar field theory, given by Eqs. (3.16), (3.18), (3.22). Noting from Eqs. (2.54) and (3.26) that $G_0(0) = I(m,T)$, we obtain to $\mathcal{O}(\lambda)$ that

$$f(T) = J(m,T) + \frac{3}{4}\,\lambda[I(m,T)]^2\,. \tag{3.45}$$

According to Eqs. (2.72), (2.73),

$$J(m,T) = -\frac{m^4}{64\pi^2}\mu^{-2\epsilon}\left[\frac{1}{\epsilon} + \ln\frac{\bar\mu^2}{m^2} + \frac{3}{2} + \mathcal{O}(\epsilon)\right] + J_T(m)\,, \tag{3.46}$$

$$I(m,T) = -\frac{m^2}{16\pi^2}\mu^{-2\epsilon}\left[\frac{1}{\epsilon} + \ln\frac{\bar\mu^2}{m^2} + 1 + \mathcal{O}(\epsilon)\right] + I_T(m)\,, \tag{3.47}$$

where the functions $J_T(m)$ and $I_T(m)$ are finite, and were evaluated in various limits in Eqs. (2.78), (2.79), (2.81), (2.92).

Inserting Eqs. (3.46), (3.47) into Eq. (3.45), we note that the result is, in general, *ultraviolet divergent*. For instance, restricting for simplicity to very high temperatures, $T \gg m$, and making use of Eq. (2.92),

$$I_T(m) \approx \frac{T^2}{12} - \frac{mT}{4\pi} + \mathcal{O}(m^2)\,, \tag{3.48}$$

the dominant term at $\epsilon \to 0$ reads

$$f(T) \approx -\frac{\mu^{-2\epsilon}}{64\pi^2\epsilon}\left\{m^4 + \lambda\left[\frac{1}{2}\,T^2m^2 - \frac{3}{2\pi}Tm^3 + \mathcal{O}(m^4)\right] + \mathcal{O}(\lambda^2)\right\} + \mathcal{O}(1)\,. \tag{3.49}$$

This result is obviously non-sensical: the divergences appear even to depend on the temperature. A proper procedure requires *renormalization*; we return to this in Sec. 3.7, but identify first also another problem with the naive result.

3.5. Naive free energy density to $\mathcal{O}(\lambda^2)$: infrared divergences

Let us consider the second order contribution to Eq. (3.45), given in Eq. (3.22). With the notation of Eqs. (3.20), (3.37), we can write it as

$$f_{(2)}(T) = -\frac{3}{4}\,\lambda^2 \int_x [G_0(x)]^4 - \frac{9}{4}\,\lambda^2[I(m,T)]^2 \int_x [G_0(x)]^2\,. \tag{3.50}$$

The question we would like to answer is, what happens if we take the limit that the particle mass m is very small, $m \ll T$. As Eqs. (3.45), (2.81), (3.48) show, at $\mathcal{O}(\lambda)$ this limit is perfectly well-defined. Consider then the first term in Eq. (3.50). We know from Eq. (3.42) that the behaviour of $G_0(x)$ is independent of m at small x; thus nothing particular happens for $|\mathbf{x}| \ll T^{-1}$. On the other hand, for large $|\mathbf{x}|$, $G_0(x)$ is given by Eq. (3.42). We can estimate the contribution as

$$\int_{|\mathbf{x}| \gtrsim T^{-1}} [G_0(x)]^4 \sim \int_0^\beta \mathrm{d}\tau \int_{|\mathbf{x}| \gtrsim T^{-1}} \mathrm{d}^3\mathbf{x}\left(\frac{Te^{-m|\mathbf{x}|}}{4\pi|\mathbf{x}|}\right)^4\,. \tag{3.51}$$

This integral is convergent even for $m \to 0$; so in the first term of Eq. (3.50), taking $m \to 0$ does not cause any divergences.

Consider then the second term in Eq. (3.50). Repeating the argument of Eq. (3.51), we get

$$\int_{|\mathbf{x}| \gtrsim T^{-1}} [G_0(x)]^2 \sim \int_0^\beta \mathrm{d}\tau \int_{|\mathbf{x}| \gtrsim T^{-1}} \mathrm{d}^3\mathbf{x}\left(\frac{Te^{-m|\mathbf{x}|}}{4\pi|\mathbf{x}|}\right)^2\,. \tag{3.52}$$

If we now attempt to take $m \to 0$, the integral is linearly divergent! Because the problem emerges from large distances, we call this an *infrared divergence.*

In fact, it is easy to be more precise about the form of the divergence. We can write

$$\begin{aligned}
\int_x [G_0(x)]^2 &= \int_x \SumInt_{\tilde{P}} \frac{e^{i\tilde{P}\cdot x}}{\tilde{P}^2+m^2} \SumInt_{\tilde{Q}} \frac{e^{i\tilde{Q}\cdot x}}{\tilde{Q}^2+m^2} \\
&= \SumInt_{\tilde{P},\tilde{Q}} \hat{\delta}(\tilde{P}+\tilde{Q}) \frac{1}{(\tilde{P}^2+m^2)(\tilde{Q}^2+m^2)} \\
&= \SumInt_{P} \frac{1}{[\tilde{P}^2+m^2]^2} \\
&= -\frac{\mathrm{d}}{\mathrm{d}m^2} I(m,T) \ . \qquad (3.53)
\end{aligned}$$

Inserting Eq. (3.48), we get

$$\int_x [G_0(x)]^2 = -\frac{1}{2m}\frac{\mathrm{d}}{\mathrm{d}m} I(m,T) = \frac{T}{8\pi m} + \mathcal{O}(1) \ . \qquad (3.54)$$

Therefore, for $m \ll T$, Eq. (3.50) evaluates to

$$f_{(2)}(T) = -\frac{9}{4}\,\lambda^2 \frac{T^4}{144}\frac{T}{8\pi m} + \mathcal{O}(m^0) \ , \qquad (3.55)$$

and indeed diverges for $m \to 0$.

It is clear that, like the ultraviolet divergence in Eq. (3.49), the infrared divergence in Eq. (3.55) must also be an artifact of some sort: a gas of weakly interacting massless scalar particles should certainly have a finite pressure and other thermodynamic properties, just like a gas of massless photons has. We return to the resolution of this problem in Sec. 3.8.

3.6. Exercise 4

(a) Let us consider a scalar field theory where the interaction term $\frac{1}{4}\,\lambda\phi^4$ is replaced with $\frac{1}{3}\,\gamma\phi^3$. Derive the expression for $f(T)$ up to $\mathcal{O}(\gamma^3)$.

(b) What kind of infrared divergences does this expression have, if we take the limit $m \to 0$?

Solution to Exercise 4

(a) According to Eq. (3.10), the radiative corrections to the free energy density can be compactly represented by the formula

$$\begin{aligned}
f_{(\geq 1)}(T) &= -\frac{1}{\beta V}\Big\langle \exp(-S_I) \Big\rangle_{0,\text{connected}} \\
&= \Big\langle S_I - \frac{1}{2}\,S_I^2 + \ldots \Big\rangle_{0,\text{connected,drop overall }\int_x} \ . \qquad (3.56)
\end{aligned}$$

The term of $\mathcal{O}(\gamma)$ obviously vanishes, since it contains an odd number of fields. The term of $\mathcal{O}(\gamma^2)$ reads

$$f_{(2)}(T) = -\frac{\gamma^2}{18}\int_x \langle \phi(x)\phi(x)\phi(x)\phi(0)\phi(0)\phi(0)\rangle_{0,\text{c}} \ . \qquad (3.57)$$

Here, according to the Wick theorem,

$$
\begin{aligned}
& \langle\phi(x)\phi(x)\phi(x)\,\phi(0)\phi(0)\phi(0)\rangle_{0,\mathrm{c}} \\
= & \; 3\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(x)\phi(x)\,\phi(0)\phi(0)\rangle_{0,\mathrm{c}} + \\
+ & \; 2\,\langle\phi(x)\phi(x)\rangle_0\,\langle\phi(x)\,\phi(0)\phi(0)\phi(0)\rangle_{0,\mathrm{c}} \\
= & \; 3\times 2\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(x)\phi(0)\rangle_0 + \\
+ & \; 3\times 1\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(x)\phi(x)\rangle_0\,\langle\phi(0)\phi(0)\rangle_0 + \\
+ & \; 2\times 3\,\langle\phi(x)\phi(x)\rangle_0\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(0)\phi(0)\rangle_0 \\
= & \; 6\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(x)\phi(0)\rangle_0 + \\
+ & \; 9\,\langle\phi(x)\phi(x)\rangle_0\,\langle\phi(x)\phi(0)\rangle_0\,\langle\phi(0)\phi(0)\rangle_0\;, \qquad (3.58)
\end{aligned}
$$

and we get

$$
f_{(2)}(T) = -\frac{1}{3}\,\gamma^2 \int_x [G_0(x)]^3 - \frac{1}{2}\,\gamma^2 [G_0(0)]^2 \int_x G_0(x)\;. \qquad (3.59)
$$

It turns out, however, that the latter term in Eq. (3.59) should actually be neglected. The reason is that once we add the interaction $\frac{1}{3}\gamma\phi^3$, then the "ground state" of the theory is no longer at $\langle\phi\rangle = 0$, as we have (implicitly) assumed; to see this, just compute $\langle\phi\rangle$ to $\mathcal{O}(\gamma)$! On the other hand, the problem can be rectified by also adding another term to the potential $V(\phi)$, of the type $\alpha\phi$. Then there are additional contributions both to $\langle\phi\rangle$ and to $f(T)$:

$$
\begin{aligned}
\langle\phi(0)\rangle & \approx \langle\phi(0)\int_x [-\alpha\phi(x) - \frac{1}{3}\,\gamma\phi(x)\phi(x)\phi(x)]\rangle_0 \\
& = -[\alpha + \gamma G_0(0)] \int_x G_0(x)\;, \qquad (3.60) \\
\delta f(T) & \approx -\frac{1}{2}\int_x \Big\langle \alpha^2\phi(x)\;\phi(0) + \frac{1}{3}\,\alpha\gamma[\phi(x)\phi(x)\phi(x)\;\phi(0) + \phi(x)\;\phi(0)\phi(0)\phi(0)]\Big\rangle_0 \\
& = -\frac{1}{2}\,[\alpha^2 + 2\alpha\gamma G_0(0)] \int_x G_0(x)\;. \qquad (3.61)
\end{aligned}
$$

Summing together Eq. (3.59) and (3.61), we get

$$
f_{(2)}(T) = -\frac{1}{3}\,\gamma^2 \int_x [G_0(x)]^3 - \frac{1}{2}\,[\alpha + \gamma G_0(0)]^2 \int_x G_0(x)\;. \qquad (3.62)
$$

Hence, we note that if we choose the coefficient α such as to make the expression in Eq. (3.60) vanish, as it should, we simultaneously need to remove part of the terms in $f_{(2)}(T)$.

To summarise, a physically meaningful expression is given by

$$
f(T) = J(m,T) - \frac{1}{3}\,\gamma^2 \int_x [G_0(x)]^3 + \mathcal{O}(\gamma^4)\;. \qquad (3.63)
$$

(b) Inspecting the large-$|\mathbf{x}|$ behaviour like in Eqs. (3.51), (3.52), we obtain

$$
\int_{|\mathbf{x}|\gtrsim T^{-1}} [G_0(x)]^3 \sim \int_0^\beta \mathrm{d}\tau \int_{|\mathbf{x}|\gtrsim T^{-1}} \mathrm{d}^3\mathbf{x} \bigg(\frac{T e^{-m|\mathbf{x}|}}{4\pi|\mathbf{x}|}\bigg)^3\;. \qquad (3.64)
$$

The radial part of the integration yields

$$
\begin{aligned}
\int_{x_0}^\infty \frac{\mathrm{d}x}{x} e^{-3mx} & = \int_{x_0}^\infty \mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}x}\ln x\right) e^{-3mx} \\
& = \Big[\ln x\, e^{-3mx}\Big]_{x_0}^\infty + 3m \int_{x_0}^\infty \mathrm{d}x\;\ln x\, e^{-3mx}
\end{aligned}
$$

$$
\begin{aligned}
&= -\ln x_0 e^{-3mx_0} + 3m\int_0^\infty \mathrm{d}x\ \ln x\, e^{-3mx} + \mathcal{O}(x_0 m) \\
&= -\ln x_0 + \int_0^\infty \mathrm{d}\hat{x}\ \ln\frac{\hat{x}}{3m}\, e^{-\hat{x}} + \mathcal{O}(x_0 m) \\
&= -\ln x_0 - \ln(3m)\int_0^\infty \mathrm{d}\hat{x}\, e^{-\hat{x}} + \int_0^\infty \mathrm{d}\hat{x}\ \ln\hat{x}\, e^{-\hat{x}} + \mathcal{O}(x_0 m) \\
&= \ln\frac{1}{3x_0 m} - \gamma_E + \mathcal{O}(x_0 m)\ ,
\end{aligned}
\tag{3.65}
$$

where we denoted $x \equiv |\mathbf{x}|$; inserted a lower bound x_0 to the integration, assuming $x_0 \ll 1/m$; and set $x_0 \to 0$ whenever that does not lead to an ultraviolet divergence.

We now see that Eq. (3.65) diverges *logarithmically* if we set $m \to 0$.