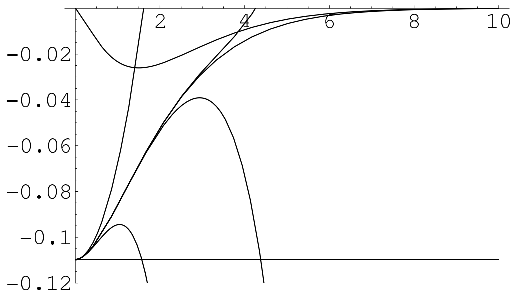
thermal_exe3.nb 1

```
full[y_] := 1 / 2 / Pi^2 NIntegrate[x^2 Log[1 - Exp[-Sqrt[x^2 + y^2]]], {x, 0, Infinity}]

lowT[y_] := -Exp[-y] (y / 2 / Pi) ^ {3 / 2}

high1[y_] := -Pi^2 / 90
high2[y_] := -Pi^2 / 90 + y^2 / 24
high3[y_] := -Pi^2 / 90 + y^2 / 24 - y^3 / 12 / Pi
high4[y_] :=
 -Pi^2 / 90 + y^2 / 24 - y^3 / 12 / Pi - y^4 / 32 / Pi^2 (Log[y] + EulerGamma - Log[4 Pi] - 3 / 4)
high5[y_] := -Pi^2 / 90 + y^2 / 24 - y^3 / 12 / Pi -
  y^4 / 32 / Pi^2 (Log[y] + EulerGamma - Log[4 Pi] - 3 / 4) + y^6 Zeta[3] / 768 / Pi^4

Plot[{full[y], lowT[y], high1[y], high2[y], high3[y], high4[y], high5[y]},
 {y, 0.01, 10}, PlotRange → {-0.12, 0}]
```



- Graphics -

Figure 1: The exact result from Eq. (2.116); the low-temperature approximation from Eq. (2.117); and five orders of the high-temperature approximation from Eq. (2.118).

3. Interacting scalar fields

In order to move from a free to an interacting theory, we now choose

$$V(\phi) \equiv \frac{1}{2}\, m^2\phi^2 + \frac{1}{4}\, \lambda\phi^4 \tag{3.1}$$

in Eq. (2.4), where $\lambda > 0$ is a dimensionless coupling constant. Thereby the Minkowskian and Euclidean Lagrangians become

$$\begin{aligned} \mathcal{L}_M &= \frac{1}{2}\, \partial^\mu\phi\partial_\mu\phi - \frac{1}{2}\, m^2\phi^2 - \frac{1}{4}\, \lambda\phi^4 \;, && (3.2) \\ \mathcal{L}_E &= \frac{1}{2}\, \partial_\mu\phi\partial_\mu\phi + \frac{1}{2}\, m^2\phi^2 + \frac{1}{4}\, \lambda\phi^4 \;, && (3.3) \end{aligned}$$

where repeated indices are again summed over, irrespective of whether they are up and down, or both at the same altitude; and the case with all indices down implies the use of Euclidean metric (i.e. no minus signs), like in Eq. (2.7).

Now, in the presence of $\lambda > 0$, it is no longer possible to exactly determine the partition function of the system. We therefore need to develop approximation methods, which could in principle be either analytic or numerical. In the following we restrict our attention to the simplest analytic procedure which, as we will see, already teaches us quite a lot about the nature of the system.

3.1. Weak-coupling expansion

In the weak-coupling expansion the theory is solved by assuming that $\lambda \ll 1$, and by expressing the result for the physical observable in question as a (generalized) Taylor series in λ.

The physical observable we are interested in, is the partition function in Eq. (2.6). Defining the free and interacting parts of the Euclidean action as

$$S_0 \equiv \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}\,\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}\,m^2\phi^2\right], \tag{3.4}$$

$$S_I \equiv \lambda \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \left[\frac{1}{4}\,\phi^4\right], \tag{3.5}$$

the partition function can be written as

$$\begin{aligned}
\mathcal{Z}^{\mathrm{SFT}}(T) &= C \int \mathcal{D}\phi\, e^{-S_0-S_I} \\
&= C \int \mathcal{D}\phi\, e^{-S_0} \left[1 - S_I + \frac{1}{2}\,S_I^2 - \frac{1}{6}\,S_I^3 + \ldots\right] \\
&= \mathcal{Z}^{\mathrm{SFT}}_{(0)} \left[1 - \langle S_I\rangle_0 + \frac{1}{2}\,\langle S_I^2\rangle_0 - \frac{1}{6}\,\langle S_I^3\rangle_0 + \ldots\right],
\end{aligned} \tag{3.6}$$

where

$$\mathcal{Z}^{\mathrm{SFT}}_{(0)} = C \int \mathcal{D}\phi\, e^{-S_0} \tag{3.7}$$

is the free partition function that we determined in Sec. 2, and the expectation value $\langle\cdots\rangle_0$ is defined as

$$\langle\cdots\rangle_0 \equiv \frac{C\int\mathcal{D}\phi\,[\cdots]\exp(-S_0)}{C\int\mathcal{D}\phi\,\exp(-S_0)}\,. \tag{3.8}$$

The free energy density then reads

$$\begin{aligned}
\frac{F^{\mathrm{SFT}}(T,V)}{V} &= -\frac{T}{V}\ln\mathcal{Z}^{\mathrm{SFT}} \\
&= \frac{F^{\mathrm{SFT}}_{(0)}}{V} - \frac{T}{V}\ln\left[1 - \langle S_I\rangle_0 + \frac{1}{2}\,\langle S_I^2\rangle_0 - \frac{1}{6}\,\langle S_I^3\rangle_0 + \ldots\right] \qquad (3.9) \\
&= \frac{F^{\mathrm{SFT}}_{(0)}}{V} - \frac{T}{V}\bigg\{-\big\langle S_I\big\rangle_0 + \frac{1}{2}\left[\big\langle S_I^2\big\rangle_0 - \big\langle S_I\big\rangle_0^2\right] - \\
&\qquad -\frac{1}{6}\left[\big\langle S_I^3\big\rangle_0 - 3\big\langle S_I\big\rangle_0\big\langle S_I^2\big\rangle_0 + 2\big\langle S_I\big\rangle_0^3\right] + \ldots\bigg\}, \qquad (3.10)
\end{aligned}$$

where we have Taylor-expanded the logarithm, $\ln(1-x) = -x - x^2/2 - x^3/3 + \ldots$. The first term, $F^{\mathrm{SFT}}_{(0)}/V$, is given in Eq. (2.26), while the subsequent terms correspond to corrections of orders $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$, and $\mathcal{O}(\lambda^3)$. As we will see, the combinations that appear within the square brackets in Eq. (3.10) have a very specific significance: Eq. (3.10) is actually *simpler* than Eq. (3.9)!

Inserting Eq. (3.5) into the various terms in Eq. (3.10), we are lead to evaluate expectation values of the type

$$\langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle_0\,. \tag{3.11}$$

These can be reduced to products of free two-point correlators, $\langle\phi(x_k)\phi(x_l)\rangle_0$, through the Wick theorem, as we now recall.

3.2. Wick theorem

The Wick theorem states that free (Gaussian) expectation values of any number of integration variables can be reduced to products of two-point correlators, according to

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_{n-1})\phi(x_n)\rangle_0 = \sum_{\text{all combinations}} \langle \phi(x_1)\phi(x_2)\rangle_0 \cdots \langle \phi(x_{n-1})\phi(x_n)\rangle_0 \; . \tag{3.12}$$

Before applying this to the terms in Eq. (3.10), we briefly recall, for completeness, how the theorem can be derived with (path) integration techniques.

Let us assume that we discretise the space-time such that the coordinates x only take a finite number of values (provided that the volume is finite as well). Then we can collect the values $\phi(x), \forall x$, into a single vector v. The free action can be written as $S_0 = \frac{1}{2}\, v^T A v$, where A is a matrix; we assume that A^{-1} exists and that A is symmetric, $A^T = A$.

The trick allowing to evaluate various integrals weighted by $\exp(-S_0)$ is to introduce a source vector b, and to take derivatives with respect to its components. Specifically, we introduce

$$\begin{aligned} \exp\Big[W(b)\Big] \quad &\equiv \quad \int \mathrm{d}v \, \exp\Big[-\frac{1}{2}\, v_i A_{ij} v_j + b_i v_i\Big] \\ &\overset{v_i \to v_i + A^{-1}_{ij} b_j}{=} \quad \exp\Big[\frac{1}{2}\, b_i A^{-1}_{ij} b_j\Big] \int \mathrm{d}v \, \exp\Big[-\frac{1}{2}\, v_i A_{ij} v_j\Big] \; , \end{aligned} \tag{3.13}$$

where we made use of a substitution of integration variables. We then obtain

$$\begin{aligned} \langle v_k v_l \ldots v_n \rangle_0 \quad &= \quad \frac{\int \mathrm{d}v \, (v_k v_l \ldots v_n) \exp\Big[-\frac{1}{2}\, v_i A_{ij} v_j\Big]}{\int \mathrm{d}v \, \exp\Big[-\frac{1}{2}\, v_i A_{ij} v_j\Big]} \\ &= \quad \frac{\Big\{ \frac{\mathrm{d}}{\mathrm{d}b_k} \frac{\mathrm{d}}{\mathrm{d}b_l} \cdots \frac{\mathrm{d}}{\mathrm{d}b_n} \exp\big[W(b)\big] \Big\}_{b=0}}{\exp\big[W(0)\big]} \\ &= \quad \Big\{ \frac{\mathrm{d}}{\mathrm{d}b_k} \frac{\mathrm{d}}{\mathrm{d}b_l} \cdots \frac{\mathrm{d}}{\mathrm{d}b_n} \exp\Big[\frac{1}{2}\, b_i A^{-1}_{ij} b_j\Big] \Big\}_{b=0} \\ &= \quad \Big\{ \frac{\mathrm{d}}{\mathrm{d}b_k} \frac{\mathrm{d}}{\mathrm{d}b_l} \cdots \frac{\mathrm{d}}{\mathrm{d}b_n} \Big[1 + \frac{1}{2}\, b_i A^{-1}_{ij} b_j + \frac{1}{2}\Big(\frac{1}{2}\Big)^2 b_i A^{-1}_{ij} b_j b_r A^{-1}_{rs} b_s + \ldots\Big] \Big\}_{b=0} \; . \end{aligned} \tag{3.14}$$

Taking the derivatives in Eq. (3.14), it is clear that:

- $\langle 1 \rangle_0 = 1$.
- If there is an odd number of components of v in the expectation value, the result is zero.
- $\langle v_k v_l \rangle_0 = A^{-1}_{kl}$.
- $\begin{aligned}\langle v_k v_l v_m v_n \rangle_0 &= A^{-1}_{kl} A^{-1}_{mn} + A^{-1}_{km} A^{-1}_{ln} + A^{-1}_{kn} A^{-1}_{lm} \\ &= \langle v_k v_l \rangle_0 \langle v_m v_n \rangle_0 + \langle v_k v_m \rangle_0 \langle v_l v_n \rangle_0 + \langle v_k v_n \rangle_0 \langle v_l v_m \rangle_0 \; .\end{aligned}$
- And that, in general, we are lead to a discretized version of Eq. (3.12).

Since all of the operations carried out are purely combinatorial in nature, it is clear that removing the discretization does not modify the result, and that thereby Eq. (3.12) indeed also holds in the continuum limit.

Let us now apply Eq. (3.12) to Eq. (3.10). We will denote

$$f(T) \equiv \lim_{V\to\infty} \frac{F(T,V)}{V} \;, \tag{3.15}$$

where we have dropped out the superscript "SFT" for simplicity. From Eqs. (2.26), (2.51), (3.10), the leading order result is the familiar one,

$$f_{(0)}(T) = J(m,T) \;. \tag{3.16}$$

At the first order, we get

$$f_{(1)}(T) = \lim_{V\to\infty} \frac{T}{V} \langle S_I \rangle_0 = \lim_{V\to\infty} \frac{T}{V} \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x}\, \frac{\lambda}{4} \langle \phi(x)\phi(x)\phi(x)\phi(x)\rangle_0 \;. \tag{3.17}$$

Here we can use the Wick theorem, Eq. (3.12). Noting furthermore that $\langle \phi(x)\phi(y)\rangle_0$ can only depend on $x-y$, due to translational invariance, the space-time integral is trivial, and we get

$$f_{(1)}(T) = \frac{3}{4}\,\lambda \langle \phi(0)\phi(0)\rangle_0 \langle \phi(0)\phi(0)\rangle_0 \;. \tag{3.18}$$

Finally, at the second order, we get

$$\begin{aligned} f_{(2)}(T) &= \lim_{V\to\infty} \left\{ -\frac{T}{2V} \Big[\langle S_I^2\rangle_0 - \langle S_I \rangle_0^2 \Big] \right\} \\ &= \lim_{V\to\infty} \left\{ -\frac{T}{2V} \bigg[\int_{x,y} \Big(\frac{\lambda}{4}\Big)^2 \langle \phi(x)\phi(x)\phi(x)\phi(x)\, \phi(y)\phi(y)\phi(y)\phi(y)\rangle_0 - \right. \\ &\qquad \left. - \int_x \frac{\lambda}{4} \langle \phi(x)\phi(x)\phi(x)\phi(x)\rangle_0 \int_y \frac{\lambda}{4} \langle \phi(y)\phi(y)\phi(y)\phi(y)\rangle_0 \bigg] \right\} \;, \end{aligned} \tag{3.19}$$

where we have denoted

$$\int_x \equiv \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \;. \tag{3.20}$$

Carrying out contractions according to Eq. (3.12), the role of the "subtraction term", the second one in Eq. (3.19), now becomes clear: it cancels all *disconnected* contractions, i.e. contractions of the type where all fields at point x are contracted with other fields at the same point. In other words, the combination in Eq. (3.19) amounts to taking into account only the *connected* contractions. This miraculous combinatorial fact is brought about by the logarithm in Eq. (3.10), i.e., by going from the partition function to the free energy!

As far as the connected contractions are concerned, the Wick theorem tells that

$$\begin{aligned} &\langle \phi(x)\phi(x)\phi(x)\phi(x)\, \phi(y)\phi(y)\phi(y)\phi(y)\rangle_{0,\mathrm{c}} \\ &\quad = 4\,\langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(x)\phi(x)\, \phi(y)\phi(y)\phi(y)\rangle_{0,\mathrm{c}} + \\ &\quad + 3\,\langle \phi(x)\phi(x)\rangle_0\, \langle \phi(x)\phi(x)\, \phi(y)\phi(y)\phi(y)\phi(y)\rangle_{0,\mathrm{c}} \\ &\quad = 4\times 3\,\langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(x)\, \phi(y)\phi(y)\rangle_{0,\mathrm{c}} + \\ &\quad + 4\times 2\,\langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(x)\rangle_0\, \langle \phi(x)\, \phi(y)\phi(y)\phi(y)\rangle_{0,\mathrm{c}} + \\ &\quad + 3\times 4\,\langle \phi(x)\phi(x)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\, \phi(y)\phi(y)\phi(y)\rangle_{0,\mathrm{c}} \\ &\quad = 4\times 3\times 2\,\langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0 + \\ &\quad + (4\times 3 + 4\times 2\times 3 + 3\times 4\times 3)\langle \phi(x)\phi(x)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0\, \langle \phi(x)\phi(y)\rangle_0\, \langle \phi(y)\phi(y)\rangle_0 \;, \end{aligned} \tag{3.21}$$

where the subscript $(...)_\mathrm{c}$ refers to "connected".

Inspecting the two-point correlators in Eq. (3.21), we note that they either depend on $x - y$, or on neither x nor y (in the cases where fields at the same point are contracted). Thereby one of the spacetime integrals is again trivial (just substitute $x \to x + y$, and note that $\langle\phi(x+y)\phi(y)\rangle_0 = \langle\phi(x)\phi(0)\rangle_0$), and cancels against the factor $T/V = 1/(\beta V)$ in Eq. (3.19). In total, then

$$f_{(2)}(T) = -\Big(\frac{\lambda}{4}\Big)^2 \bigg[12 \int_x \langle\phi(x)\phi(0)\rangle_0^4 + 36\langle\phi(0)\phi(0)\rangle_0^2 \int_x \langle\phi(x)\phi(0)\rangle_0^2\bigg] . \tag{3.22}$$

One could go on with the third-order terms in Eq. (3.10): again, it could be verified that the "subtraction terms" cancel all disconnected contractions, so that only the connected ones contribute to $f(T)$; and that one spacetime integral cancels against the explicit factor T/V. These facts are, in fact, of general nature, and hold at any order in the weak-coupling expansion.

To summarise, the Wick theorem has allowed us to reduce the terms in Eq. (3.10) to various structures made of the two-point correlator $\langle\phi(x)\phi(0)\rangle_0$. We now turn to its properties.

3.3. Propagator

The two-point correlator $\langle\phi(x)\phi(y)\rangle_0$ is usually called the *free propagator.* Denoting

$$\bar{\delta}(\tilde{P} + \tilde{Q}) \equiv \int_x e^{i(\tilde{P}+\tilde{Q})\cdot x} = \beta\delta_{\tilde{p}_n+\tilde{q}_n,0}(2\pi)^d\delta^{(d)}(\mathbf{p}+\mathbf{q}) , \tag{3.23}$$

where $\tilde{P} \equiv (\tilde{p}_n, \mathbf{p})$, and $\tilde{p}_n$ are bosonic Matsubara frequencies; and employing the Fourier-space presentation

$$\phi(x) \equiv \sum\!\!\!\!\!\!\!\int_{\tilde{P}} \tilde{\phi}(\tilde{P})e^{i\tilde{P}\cdot x} , \tag{3.24}$$

we recall from basic quantum field theory that the propagator can be written as

$$\langle\tilde{\phi}(\tilde{P})\tilde{\phi}(\tilde{Q})\rangle_0 = \bar{\delta}(\tilde{P}+\tilde{Q})\frac{1}{\tilde{P}^2+m^2} , \tag{3.25}$$

$$\langle\phi(x)\phi(y)\rangle_0 = \sum\!\!\!\!\!\!\!\int_{\tilde{P}} e^{i\tilde{P}\cdot(x-y)}\frac{1}{\tilde{P}^2+m^2} . \tag{3.26}$$

Before inserting these expressions into Eqs. (3.18), (3.22), we briefly review their derivation, as well as some basic properties of $\langle\phi(x)\phi(y)\rangle_0$.

In order to carry out the derivation, we return for a moment to a finite volume V, and proceed as in Sec. 2.2. Let us start by inserting Eq. (3.24) into the definition of the propagator,

$$\langle\phi(x)\phi(y)\rangle_0 = \sum\!\!\!\!\!\!\!\int_{\tilde{P},\tilde{Q}} e^{i\tilde{P}\cdot x+i\tilde{Q}\cdot y}\langle\tilde{\phi}(\tilde{P})\tilde{\phi}(\tilde{Q})\rangle_0 . \tag{3.27}$$

In order to compute this expectation value, we insert Eq. (3.24) also into the free action, S_0, finding

$$S_0 = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}} \tilde{\phi}(-\tilde{P})(\tilde{P}^2+m^2)\tilde{\phi}(\tilde{P}) = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\tilde{P}} (\tilde{P}^2+m^2)|\tilde{\phi}(\tilde{P})|^2 . \tag{3.28}$$

We write $\tilde{\phi}(\tilde{P}) = a(\tilde{P}) + i\,b(\tilde{P})$, with $a(-\tilde{P}) = a(\tilde{P})$, $b(-\tilde{P}) = -b(\tilde{P})$; only half of the Fourier components are independent, and we could choose the ones specified in Eq. (2.13).

Restricting the sum to independent components, and making use of the symmetry properties of $a(\tilde{P})$ and $b(\tilde{P})$, Eq. (3.28) becomes

$$S_0 = \frac{T}{V}\sum_{\tilde{P}_{\text{indep.}}} (\tilde{P}^2+m^2)[a^2(\tilde{P}) + b^2(\tilde{P})] . \tag{3.29}$$

The Gaussian integral,

$$\frac{\int \mathrm{d}x\, x^2 \exp(-cx^2)}{\int \mathrm{d}x\, \exp(-cx^2)} = \frac{1}{2c}\;, \tag{3.30}$$

and the symmetries of $a(\tilde{P})$ and $b(\tilde{P})$, thus imply that

$$\langle a(\tilde{P})\, b(\tilde{Q})\rangle_0 \;=\; 0\;, \tag{3.31}$$

$$\langle a(\tilde{P})\, a(\tilde{Q})\rangle_0 \;=\; (\delta_{\tilde{P},\tilde{Q}} + \delta_{\tilde{P},-\tilde{Q}})\frac{V}{2T}\frac{1}{\tilde{P}^2+m^2}\;, \tag{3.32}$$

$$\langle b(\tilde{P})\, b(\tilde{Q})\rangle_0 \;=\; (\delta_{\tilde{P},\tilde{Q}} - \delta_{\tilde{P},-\tilde{Q}})\frac{V}{2T}\frac{1}{\tilde{P}^2+m^2}\;, \tag{3.33}$$

where the delta-functions are of Kronecker-type in finite volume. Thereby the propagator becomes

$$\begin{aligned}\langle\tilde{\phi}(\tilde{P})\tilde{\phi}(\tilde{Q})\rangle_0 &= \langle a(\tilde{P})\, a(\tilde{Q}) + i\, a(\tilde{P})\, b(\tilde{Q}) + i\, b(\tilde{P})\, a(\tilde{Q}) - b(\tilde{P})\, b(\tilde{Q})\rangle_0 \\ &= \delta_{\tilde{P},-\tilde{Q}}\frac{V}{T}\frac{1}{\tilde{P}^2+m^2} = \beta\delta_{\tilde{p}_n+\tilde{q}_n,0}V\delta_{\mathbf{p}+\mathbf{q},\mathbf{0}}\frac{1}{\tilde{P}^2+m^2}\;.\end{aligned} \tag{3.34}$$

In the infinite-volume limit,

$$\frac{1}{V}\sum_{\mathbf{p}} \longrightarrow \int\frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}\;, \quad V\delta_{\mathbf{p},\mathbf{0}} \longrightarrow (2\pi)^d\delta^{(d)}(\mathbf{p})\;, \tag{3.35}$$

and we recover Eq. (3.25) which, in combination with Eq. (3.27), in turn leads to Eq. (3.26).

We would now like to learn something more about the behaviour of the propagator $\langle\phi(x)\phi(y)\rangle_0$, at short and large separations $x-y$. We note, first of all, that as shown by Eqs. (1.84), (1.73),

$$T\sum_{\tilde{p}_n}\frac{e^{i\tilde{p}_n\tau}}{\tilde{p}_n^2+E^2} = \frac{1}{2E}\frac{\cosh\left[\left(\frac{\beta}{2}-\tau\right)E\right]}{\sinh\left[\frac{\beta E}{2}\right]}\;. \tag{3.36}$$

This equation is valid for $0\le\tau\le\beta$; as is obvious from the left-hand side, we can extend the validity to $-\beta\le\tau\le\beta$ by replacing τ by $|\tau|$. Thereby the propagator from Eq. (3.26) now becomes

$$G_0(x-y) \equiv \langle\phi(x)\phi(y)\rangle_0 = \int\frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\;\frac{1}{2E_\mathbf{p}}\frac{\cosh\left[\left(\frac{\beta}{2}-|x_0-y_0|\right)E_\mathbf{p}\right]}{\sinh\left[\frac{\beta E_\mathbf{p}}{2}\right]}\Bigg|_{E_\mathbf{p}\equiv\sqrt{\mathbf{p}^2+m^2}}\;. \tag{3.37}$$

Since G_0 only depends on the separation $x-y$, we set $y=0$ in the following.

Consider first short distances, $|\mathbf{x}|, |x_0| \ll 1/T, 1/m$. We may expect the dominant contribution in the Fourier transform in Eq. (3.37) to come from the regime $|\mathbf{p}||\mathbf{x}|\sim 1$, so let us assume $|\mathbf{p}|\gg T, m$. Then $E_\mathbf{p}\approx|\mathbf{p}|$, and $\beta E_\mathbf{p}\approx|\mathbf{p}|/T\gg 1$. Consequently,

$$\frac{\cosh\left[\left(\frac{\beta}{2}-|x_0|\right)E_\mathbf{p}\right]}{\sinh\left[\frac{\beta E_\mathbf{p}}{2}\right]} \approx \frac{\exp\left[\left(\frac{\beta}{2}-|x_0|\right)E_\mathbf{p}\right]}{\exp\left[\frac{\beta E_\mathbf{p}}{2}\right]} \approx e^{-|x_0||\mathbf{p}|}\;. \tag{3.38}$$

We note that

$$\frac{1}{2|\mathbf{p}|}e^{-|x_0||\mathbf{p}|} = \int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\frac{e^{ip_0x_0}}{p_0^2+\mathbf{p}^2}\;, \tag{3.39}$$

whereby

$$G_0(x) \approx \int\frac{\mathrm{d}^{d+1}P}{(2\pi)^{d+1}}\frac{e^{iP\cdot x}}{P^2}\;, \tag{3.40}$$

with $P \equiv (p_0, \mathbf{p})$.

At this point we can make use of rotational symmetry, in order to choose x in the direction of the component p_0. Then

$$\begin{aligned}
\int \frac{\mathrm{d}^{d+1}P}{(2\pi)^{d+1}} \frac{e^{iP\cdot x}}{P^2} &= \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} \frac{e^{ip_0|x|}}{p_0^2 + \mathbf{p}^2} \\
&= \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \frac{e^{-|x||\mathbf{p}|}}{2|\mathbf{p}|} \\
&= \frac{1}{(2\pi)^d} \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} \mathrm{d}|\mathbf{p}||\mathbf{p}|^{d-2} e^{-|x||\mathbf{p}|} \\
&= \frac{\Gamma(d-1)}{(4\pi)^{d/2}\Gamma(d/2)|x|^{d-1}} \,, \qquad (3.41)
\end{aligned}$$

where we made use of Eq. (2.61). Inserting $d = 3$ and $\Gamma(3/2) = \sqrt{\pi}/2$, we find

$$G_0(x) \approx \frac{1}{4\pi^2|x|^2} \,, \quad |x| \ll \frac{1}{T}, \frac{1}{m} \,. \qquad (3.42)$$

We note that this behaviour is independent of T and m: at short enough distances (in the "ultraviolet" regime), temperature and masses do not play a role, and the propagator diverges.

Consider, on the other hand, large distances, $|\mathbf{x}| \gg 1/T$ (the temporal coordinate x_0 remains by construction always "small", i.e. is at most $1/T$). We expect the Fourier-transform in Eq. (3.37) to now be dominated by small momenta, $|\mathbf{p}| \ll T$. If we simplify the situation further by assuming that we are also at very high temperatures, $T \gg m$, then $\beta E_{\mathbf{p}} \ll 1$ in the relevant regime, and we can expand the hyperbolic functions in Taylor series, $\cosh(\epsilon) \approx 1$, $\sinh(\epsilon) \approx \epsilon$. Then

$$G_0(x) \approx T \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2 + m^2} \,. \qquad (3.43)$$

Restricting for simplicity to $d = 3$, we can write[4]

$$\begin{aligned}
G_0(x) &\approx \frac{T}{(2\pi)^2} \int_{-1}^{+1} \mathrm{d}z \int_0^{\infty} \mathrm{d}|\mathbf{p}|\, |\mathbf{p}|^2 \frac{e^{i|\mathbf{p}||\mathbf{x}|z}}{|\mathbf{p}|^2 + m^2} \\
&= \frac{T}{(2\pi)^2} \int_0^{\infty} \frac{\mathrm{d}|\mathbf{p}|\, |\mathbf{p}|^2}{|\mathbf{p}|^2 + m^2} \frac{e^{i|\mathbf{p}||\mathbf{x}|} - e^{-i|\mathbf{p}||\mathbf{x}|}}{i|\mathbf{p}||\mathbf{x}|} \\
&= \frac{T}{(2\pi)^2 i|\mathbf{x}|} \int_{-\infty}^{\infty} \frac{\mathrm{d}p\, p\, e^{ip|\mathbf{x}|}}{p^2 + m^2} \\
&= T \frac{e^{-m|\mathbf{x}|}}{4\pi|\mathbf{x}|} \,, \quad |\mathbf{x}| \gg \frac{1}{T} \,, \qquad (3.44)
\end{aligned}$$

where the last integral was carried out by closing the contour in the upper half plane.

We note from Eq. (3.44) that at large distances (in the "infrared" regime), thermal effects modify the behaviour of the propagator in an essential way. In particular, if we were to set the mass to zero, then Eq. (3.42) would be the exact behaviour at zero temperature, both at small and at large distances, while Eq. (3.42) shows that a temperature would "slow down" the long-distance decay to $T/(4\pi|\mathbf{x}|)$. In other words, we can say that at a non-zero temperature the theory is more sensitive to infrared physics than at zero temperature.

[4]For a general d, $\int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\mathbf{p}^2+m^2} = \frac{1}{(2\pi)^{d/2}} (\frac{m}{|\mathbf{x}|})^{d/2-1} K_{d/2-1}(m|\mathbf{x}|)$, where K is a modified Bessel function.