

Thereby we have arrived at the main conclusion of this section: at low temperatures,  $T \ll m$ , finite-temperature effects in a theory with a mass gap are exponentially suppressed by the Boltzmann factor,  $\exp(-m/T)$ . In fact the results agree with those in non-relativistic classical statistical mechanics. Consequently, the results for the functions  $J(m, T)$ ,  $I(m, T)$  can be well approximated by their zero-temperature limits,  $J_0(m)$ ,  $I_0(m)$ , given in Eqs. (2.72), (2.73), respectively.

## 2.6. High-temperature expansion

We now move to the opposite limit than in the previous section, i.e.  $T \gg m$  or, in terms of Eq. (2.75),  $y = m/T \ll 1$ . It appears obvious that the procedure then should be a Taylor expansion in  $y^2$ , around  $y^2 = 0$ . The zeroth order term, for instance, yields

$$J_T(0) = \frac{T^4}{2\pi^2} \int_0^\infty dx x^2 \ln(1 - e^{-\sqrt{x^2}}) = -\frac{\pi^2 T^4}{90}, \quad (2.80)$$

which is nothing but the free-energy density (minus the pressure) of black-body radiation with one massless degree of freedom. A term of order  $\mathcal{O}(y^2)$  can also be computed exactly.

However, that is as far as it works: trying to go to the second order,  $\mathcal{O}(y^4)$ , one finds that the integral determining the coefficient of  $y^4$  is power-divergent for small  $x$ ! In other words, the function  $J_T(m)$  is not analytic around the origin in the variable  $m^2$ .

Nevertheless, Eq. (2.75) can still be expanded in a generalized sense, as we will see. The result will in fact read

$$J_T(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2} \left[ \ln\left(\frac{m e^{\gamma_E}}{4\pi T}\right) - \frac{3}{4} \right] + \frac{m^6 \zeta(3)}{3(4\pi)^4 T^2} + \mathcal{O}\left(\frac{m^8}{T^4}\right) + \mathcal{O}(\epsilon), \quad (2.81)$$

where  $m \equiv (m^2)^{1/2}$ . It is the cubic term in Eq. (2.81) which first indicates that  $J_T(m)$  is not analytic in  $m^2$  around the origin (because there is a branch cut); this term plays a very important role in certain physical contexts, as we will see later on.

Our goal now is to *derive* the expansion in Eq. (2.81). A classic derivation, starting directly from the definition in Eq. (2.75), can be found in a paper by Dolan and Jackiw, Phys. Rev. D 9 (1974) 3320. It will be easier, and ultimately more useful, to carry out another type of derivation, however: we start from Eq. (2.51) rather than Eq. (2.50), and now carry out *first the integration*  $\int_{\mathbf{k}}$ , and only then the sum  $\sum_{\omega_n}$ .

Of course, Eq. (2.51) contains inconvenient additional constant terms, which appears problematic. Fortunately we already know that mass-independent correct value of  $J(0, T)$ : it is given in Eq. (2.80). Therefore it is enough to study  $I(m, T)$ , in which case the starting point, Eq. (2.54), is simple enough, and subsequently integrate for  $J(m, T)$  as

$$J(m, T) = \int_0^m dm' m' I(m', T) + J(0, T). \quad (2.82)$$

Proceeding now with  $I(m, T)$  in Eq. (2.54), the essential insight is to split the sum into the contribution of the zero-mode,  $\omega_n = 0$ , and that of the non-zero modes,  $\omega_n \neq 0$ . Using the notation of Eq. (2.55), we thus write

$$\not\int_{\bar{K}} = \not\int'_{\bar{K}} + T \int_{\mathbf{k}}. \quad (2.83)$$

Let us first compute the contribution of the last term, which will be denoted by  $I^{(n=0)}$ .

To start with, let us return to the *infrared divergences* alluded to above. Trying naively a Taylor expansion in  $m^2$ , we would get

$$I^{(n=0)} = T \int_{\mathbf{k}} \frac{1}{\mathbf{k}^2 + m^2} = T \int \frac{d^d \mathbf{k}}{(2\pi)^d} \left[ \frac{1}{\mathbf{k}^2} - \frac{m^2}{\mathbf{k}^4} + \frac{m^4}{\mathbf{k}^6} + \dots \right]. \quad (2.84)$$

For  $d = 3 - 2\epsilon$ , the first term is “ultraviolet divergent”, i.e. grows at large  $|\mathbf{k}|$ ; while the second and subsequent terms are “infrared divergent”, i.e. explode at small  $|\mathbf{k}|$  too fast to be integrable. (Of course, in dimensional regularization, every term in Eq. (2.84) appears strictly speaking to be zero. The total result is non-zero, however, as we will see: thus the Taylor expansion in Eq. (2.84) is really not justified, whichever way one looks at it.)

We now compute the integral in Eq. (2.84) properly. The result can be directly read from Eq. (2.64), by just setting  $d = 3 - 2\epsilon$ ,  $A = 1$ :

$$I^{(n=0)} = TF(m, 3-2\epsilon, 1) = T \frac{1}{(4\pi)^{3/2-\epsilon}} \frac{\Gamma(-\frac{1}{2} + \epsilon)}{\Gamma(1)} \frac{1}{(m^2)^{-1/2+\epsilon}} \stackrel{\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}}{=} -\frac{Tm}{4\pi} + \mathcal{O}(\epsilon). \quad (2.85)$$

This result is quite remarkable: a linearly divergent integral over a manifestly positive function is *finite* and *negative* in dimensional regularization! According to Eq. (2.52), the corresponding term in  $J^{(n=0)}$  reads

$$J^{(n=0)} = -\frac{Tm^3}{12\pi} + \mathcal{O}(\epsilon). \quad (2.86)$$

Given the importance of the result, and its somewhat counter-intuitive appearance, it is worthwhile to demonstrate that Eq. (2.85) is *not* an artifact of dimensional regularization. Indeed, let us compute it with cutoff regularization, by restricting  $|\mathbf{k}|$  to be smaller than an upper bound,  $\Lambda$ :

$$\begin{aligned} I^{(n=0)} &= T \frac{4\pi}{(2\pi)^3} \int_0^\Lambda \frac{d\mathbf{k} \mathbf{k}^2}{\mathbf{k}^2 + m^2} \\ &= \frac{T}{2\pi^2} \left[ \Lambda - m^2 \int_0^\Lambda \frac{d\mathbf{k}}{\mathbf{k}^2 + m^2} \right] = \frac{T}{2\pi^2} \left[ \Lambda - m \arctan\left(\frac{\Lambda}{m}\right) \right] \\ &= T \left[ \frac{\Lambda}{2\pi^2} - \frac{m}{4\pi} + \mathcal{O}\left(\frac{m^2}{\Lambda}\right) \right]. \end{aligned} \quad (2.87)$$

We now observe that, due to the first term, Eq. (2.87) *is* indeed positive. This term is unphysical, however: it must be cancelled by similar terms emerging from the non-zero modes, since the temperature-dependent part of Eq. (2.53) is manifestly finite. Representing a power divergence, it does not appear in dimensional regularization at all. The second term in Eq. (2.87) is the physical one; it indeed agrees with Eq. (2.85). The remaining terms in Eq. (2.87) vanish when we take the cutoff to infinity, and are the analogy of the  $\mathcal{O}(\epsilon)$ -terms of Eq. (2.85).

We next turn to the contribution of the non-zero Matsubara modes, which will be denoted by  $I'(m, T)$ . It is important to realise that in this case, a Taylor expansion in  $m^2$  *can be carried out*: the integrals will be of the type

$$\int_{\mathbf{k}} \frac{(m^2)^n}{(\omega_n^2 + \mathbf{k}^2)^{n+1}}, \quad \omega_n \neq 0, \quad (2.88)$$

and thus the integrand remains finite for small  $|\mathbf{k}|$ , i.e., there are no infrared divergences. (There could be ultraviolet divergences at small  $n$ , but these are taken care of by the regularization.)

More explicitly,

$$I'(m, T) = T \sum_{\omega'_n} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{1}{\omega_n^2 + \mathbf{k}^2 + m^2}$$

$$\begin{aligned}
&\stackrel{\text{Taylor}}{=} 2T \sum_{n=1}^{\infty} \int \frac{d^d \mathbf{k}}{(2\pi)^d} \sum_{l=0}^{\infty} (-1)^l \frac{m^{2l}}{[(2\pi nT)^2 + \mathbf{k}^2]^{l+1}} \\
\text{Eq. (2.64)} \quad &\stackrel{=}{=} 2T \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (-1)^l m^{2l} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(l+1 - \frac{d}{2})}{\Gamma(l+1)} \frac{1}{(2\pi nT)^{2l+2-d}} \\
&= 2T \frac{1}{(4\pi)^{d/2} (2\pi T)^{2-d}} \sum_{l=0}^{\infty} \left[ \frac{-m^2}{(2\pi T)^2} \right]^l \frac{\Gamma(l+1 - \frac{d}{2})}{\Gamma(l+1)} \zeta(2l+2-d), \quad (2.89)
\end{aligned}$$

where in the last step we have interchanged the orders of the two summations, and identified the  $n$ -sum as a Riemann zeta function,  $\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}$ . Some of the properties of  $\zeta(s)$  are summarised in Sec. 2.7.

Let us work out the orders  $l = 0, 1, 2$  explicitly. For  $d = 3 - 2\epsilon$ , the order  $l = 0$  requires evaluating  $\Gamma(-\frac{1}{2} + \epsilon)$  and  $\zeta(-1 + 2\epsilon)$ ;  $l = 1$  requires evaluating  $\Gamma(\frac{1}{2} + \epsilon)$  and  $\zeta(1 + 2\epsilon)$ ; and  $l = 2$  requires evaluating  $\Gamma(\frac{3}{2} + \epsilon)$  and  $\zeta(3 + 2\epsilon)$ . We give some more details in Sec. 2.7 and in Exercise 3. In any case, a straightforward computation yields

$$I'(m, T) = \frac{T^2}{12} - \frac{2m^2}{(4\pi)^2} \mu^{-2\epsilon} \left[ \frac{1}{2\epsilon} + \ln \left( \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right) \right] + \frac{2m^4 \zeta(3)}{(4\pi)^4 T^2} + \mathcal{O} \left( \frac{m^6}{T^4} \right) + \mathcal{O}(\epsilon). \quad (2.90)$$

Adding the zero-mode contribution, Eq. (2.85), we get

$$I(m, T) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{2m^2}{(4\pi)^2} \mu^{-2\epsilon} \left[ \frac{1}{2\epsilon} + \ln \left( \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right) \right] + \frac{2m^4 \zeta(3)}{(4\pi)^4 T^2} + \mathcal{O} \left( \frac{m^6}{T^4} \right) + \mathcal{O}(\epsilon). \quad (2.91)$$

Subtracting Eq. (2.73), finally yields

$$I_T(m) = \frac{T^2}{12} - \frac{mT}{4\pi} - \frac{2m^2}{(4\pi)^2} \left[ \ln \left( \frac{m e^{\gamma_E}}{4\pi T} \right) - \frac{1}{2} \right] + \frac{2m^4 \zeta(3)}{(4\pi)^4 T^2} + \mathcal{O} \left( \frac{m^6}{T^4} \right) + \mathcal{O}(\epsilon). \quad (2.92)$$

Note how the divergences and  $\bar{\mu}$  have cancelled from  $I_T(m)$ , as must be the case.

We can now transport these results to various versions of the function  $J$ , by making use of Eqs. (2.80) and (2.82). From Eq. (2.90), we get

$$J'(m, T) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^4}{2(4\pi)^2} \left[ \frac{1}{2\epsilon} + \ln \left( \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right) \right] + \frac{m^6 \zeta(3)}{3(4\pi)^4 T^2} + \mathcal{O} \left( \frac{m^8}{T^4} \right) + \mathcal{O}(\epsilon). \quad (2.93)$$

Adding the zero-mode contribution from Eq. (2.86) leads to

$$J(m, T) = -\frac{\pi^2 T^4}{90} + \frac{m^2 T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{2(4\pi)^2} \left[ \frac{1}{2\epsilon} + \ln \left( \frac{\bar{\mu} e^{\gamma_E}}{4\pi T} \right) \right] + \frac{m^6 \zeta(3)}{3(4\pi)^4 T^2} + \mathcal{O} \left( \frac{m^8}{T^4} \right) + \mathcal{O}(\epsilon). \quad (2.94)$$

Subtracting the zero-temperature part,  $J_0(m)$  in Eq. (2.72), leads finally to the expansion for  $J_T(m)$ , given in Eq. (2.81). Note again the cancellation of  $1/\epsilon$  and  $\bar{\mu}$  from  $J_T(m)$ . The numerical convergence of the high-temperature expansion is inspected in Exercise 3.

## 2.7. Properties of the Euler gamma and Riemann zeta functions

$\Gamma(s)$

The function  $\Gamma(s)$  is to be viewed as a complex function, with a complex argument  $s$ . For  $\text{Re}(s) > 0$ , it can be defined as

$$\Gamma(s) = \int_0^{\infty} dx x^{s-1} e^{-x} , \quad (2.95)$$

while for  $\text{Re}(s) \leq 0$ , the values can be obtained through iterative use of the relation

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} . \quad (2.96)$$

On the real axis,  $\text{Im}(s) = 0$ ,  $\Gamma(s)$  is regular at  $s = 1$ ; as a consequence of Eq. (2.96), it then has first order poles at  $s = 0, -1, -2, \dots$ .

In practical applications, the argument  $s$  is typically either close to an integer, or close to a half integer. In the former case, we can use Eq. (2.96) to relate the desired value to the values of  $\Gamma(s)$  and its derivatives around  $s = 1$ ; these can then be worked out from the convergent integral representation in Eq. (2.95). In particular,

$$\Gamma(1) = 1 , \quad \Gamma'(1) = -\gamma_E , \quad (2.97)$$

where  $\gamma_E$  is the Euler constant,  $\gamma_E = 0.577215664901\dots$ . In the latter case, we can use Eq. (2.96) to relate the desired value to the value of  $\Gamma(s)$  and its derivatives around  $s = \frac{1}{2}$ ; these can then be worked out from the convergent integral representation in Eq. (2.95). In particular,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} , \quad \Gamma'\left(\frac{1}{2}\right) = \sqrt{\pi}(-\gamma_E - 2 \ln 2) . \quad (2.98)$$

The values required for Eq. (2.90) thus become

$$\Gamma\left(-\frac{1}{2} + \epsilon\right) = -2\sqrt{\pi} + \mathcal{O}(\epsilon) , \quad (2.99)$$

$$\Gamma\left(\frac{1}{2} + \epsilon\right) = \sqrt{\pi} \left[ 1 - \epsilon(\gamma_E + 2 \ln 2) + \mathcal{O}(\epsilon^2) \right] , \quad (2.100)$$

$$\Gamma\left(\frac{3}{2} + \epsilon\right) = \frac{\sqrt{\pi}}{2} + \mathcal{O}(\epsilon) . \quad (2.101)$$

We have gone one order deeper in the middle one, because the result turns out to be multiplied by  $1/\epsilon$  (cf. Eq. (2.111)).

$\zeta(s)$

The function  $\zeta(s)$  is to be viewed as a complex function, with a complex argument  $s$ . For  $\text{Re}(s) > 1$ , it can be defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{dx x^{s-1}}{e^x - 1} . \quad (2.102)$$

The equivalence of the two forms in Eq. (2.102) can be seen by writing  $1/(e^x - 1) = e^{-x}/(1 - e^{-x}) = \sum_{n=1}^{\infty} e^{-nx}$ , and using the definition of the gamma function in Eq. (2.95). The remarkable properties of  $\zeta(s)$  follow from the fact that by writing

$$\frac{1}{e^x - 1} = \frac{1}{(e^{x/2} - 1)(e^{x/2} + 1)} = \frac{1}{2} \left[ \frac{1}{e^{x/2} - 1} - \frac{1}{e^{x/2} + 1} \right] , \quad (2.103)$$

and then substituting integration variables,  $x \rightarrow 2x$ , we find an alternative integral representation for  $\zeta(s)$ ,

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{dx x^{s-1}}{e^x + 1}. \quad (2.104)$$

The integral here is defined for  $\text{Re}(s) > 0$ . Moreover, even though it diverges at  $s \rightarrow 0$ , the function  $\Gamma(s)$  also diverges at the same point, and consequently  $\zeta(0)$  will be finite and regular around  $s = 0$ :

$$\zeta(0) = -\frac{1}{2} \left[ = \left. \sum_{n=1}^{\infty} n! \right] \right], \quad (2.105)$$

$$\zeta'(0) = -\frac{1}{2} \ln(2\pi). \quad (2.106)$$

Finally, for  $\text{Re}(s) \leq 0$ , an analytic continuation is obtained through

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s). \quad (2.107)$$

On the real axis,  $\text{Im}(s) = 0$ ,  $\zeta(s)$  has a pole only at  $s = 1$ . Its values at even arguments are “easy”; in fact, at negative even integers, Eq. (2.107) implies that

$$\zeta(-2n) = 0, \quad n = 1, 2, 3, \dots, \quad (2.108)$$

while at positive even integers the values can be related to the Bernoulli numbers,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \dots \quad (2.109)$$

Negative odd integers can be related to the positive even ones through Eq. (2.107), which equation also allows to determine the behaviour around the pole at  $s = 1$ . In contrast, odd positive integers larger than unity, i.e.  $s = 3, 5, \dots$ , yield new transcendental numbers.

The values required for Eq. (2.90) become

$$\zeta(-1+2\epsilon) = -\frac{1}{2\pi^2} \Gamma(2) \zeta(2) + \mathcal{O}(\epsilon) = -\frac{1}{12} + \mathcal{O}(\epsilon), \quad (2.110)$$

$$\begin{aligned} \zeta(1+2\epsilon) &= 2^{1+2\epsilon} \pi^{2\epsilon} \left[ \sin\left(\frac{\pi}{2}\right) + \pi\epsilon \cos\left(\frac{\pi}{2}\right) \right] \left(-\frac{1}{2\epsilon}\right) \Gamma(1-2\epsilon) \zeta(-2\epsilon) \\ &= 2(1+2\epsilon \ln 2)(1+2\epsilon \ln \pi)(1) \left(-\frac{1}{2\epsilon}\right) (1+2\epsilon\gamma_E) \left(-\frac{1}{2}\right) (1-2\epsilon \ln 2\pi) + \mathcal{O}(\epsilon) \\ &= \frac{1}{2\epsilon} + \gamma_E + \mathcal{O}(\epsilon), \end{aligned} \quad (2.111)$$

$$\zeta(3+2\epsilon) = \zeta(3) + \mathcal{O}(\epsilon) \approx 1.2020569031\dots + \mathcal{O}(\epsilon), \quad (2.112)$$

where in the first two cases we made use of Eq. (2.107), and in the second also of Eqs. (2.105), (2.106).

### 2.8. Exercise 3

- (a) Complete the derivation leading to Eq. (2.90).
- (b) Inspecting  $J_T(m)$ , Eq. (2.81), sketch the regimes where the low-temperature and high-temperature expansions are numerically accurate.

#### Solution

- (a) For the term  $l = 0$  in Eq. (2.89), we make use of the values in Eqs. (2.99), (2.110):

$$I'(m, T)|_{l=0} = 2T \frac{1}{(4\pi)^{3/2}} (2\pi T)^{-2\sqrt{\pi}} \left( -\frac{1}{12} \right) + \mathcal{O}(\epsilon) = \frac{T^2}{12} + \mathcal{O}(\epsilon). \quad (2.113)$$

For the term  $l = 1$  in Eq. (2.89), we make use of the values in Eqs. (2.100), (2.111):

$$\begin{aligned} I'(m, T)|_{l=1} &= 2T \frac{(4\pi)^\epsilon}{(4\pi)^{3/2}} (2\pi T)^{1-2\epsilon} \left[ \frac{-m^2}{(2\pi T)^2} \right] \sqrt{\pi} \left[ 1 - \epsilon(\gamma_E + 2 \ln 2) \right] \frac{1}{2\epsilon} (1 + 2\epsilon\gamma_E) + \mathcal{O}(\epsilon) \\ &\stackrel{1=\mu^{-2\epsilon}\mu^{2\epsilon}}{=} \frac{m^2}{(4\pi)^2} \mu^{-2\epsilon} \left\{ \frac{1}{\epsilon} + \ln \frac{\mu^2}{T^2} + \ln(4\pi) - \gamma_E + 2[\gamma_E - \ln(4\pi)] \right\} + \mathcal{O}(\epsilon) \\ &= -\frac{m^2}{(4\pi)^2} \mu^{-2\epsilon} \left\{ \frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{T^2} + 2 \ln \left( \frac{e^{\gamma_E}}{4\pi} \right) \right\} + \mathcal{O}(\epsilon), \end{aligned} \quad (2.114)$$

where in the last step we introduced the  $\overline{\text{MS}}$  scheme scale parameter through Eq. (2.71). For the term  $l = 2$  in Eq. (2.89), we make use of the values in Eqs. (2.101), (2.112):

$$I'(m, T)|_{l=2} = 2T \frac{1}{(4\pi)^{3/2}} (2\pi T)^{\frac{m^4}{(2\pi T)^4} \frac{1}{2} \sqrt{\pi}} \zeta(3) + \mathcal{O}(\epsilon) = \frac{2m^4 \zeta(3)}{(4\pi)^4 T^2} + \mathcal{O}(\epsilon). \quad (2.115)$$

- (b) We again denote  $y \equiv m/T$ , and inspect then the function

$$\mathcal{J}(y) \equiv \frac{J_T(m)}{T^4} = \frac{1}{2\pi^2} \int_0^\infty dx x^2 \ln \left( 1 - e^{-\sqrt{x^2+y^2}} \right). \quad (2.116)$$

Apart from evaluating this expression numerically, we also consider the low-temperature result from Eq. (2.78),

$$\mathcal{J}(y) \stackrel{y \gg 1}{\approx} -\left( \frac{y}{2\pi} \right)^{\frac{3}{2}} e^{-y} + \mathcal{O}(\epsilon), \quad (2.117)$$

as well as the high-temperature result from Eq. (2.81),

$$\mathcal{J}(y) \stackrel{y \ll 1}{\approx} -\frac{\pi^2}{90} + \frac{y^2}{24} - \frac{y^3}{12\pi} - \frac{y^4}{2(4\pi)^2} \left[ \ln \left( \frac{ye^{\gamma_E}}{4\pi} \right) - \frac{3}{4} \right] + \frac{y^6 \zeta(3)}{3(4\pi)^4} + \mathcal{O}(\epsilon). \quad (2.118)$$

The results of the comparison are shown in Fig. 1. We observe that if we keep terms up to  $y^6$  in the high-temperature expansion, its numerical convergence is reasonable for  $y \lesssim 3$ . On the other hand, the low-temperature expansion converges reasonably well as soon as  $y \gtrsim 6$ . In between, a numerical evaluation is needed.