2. Free scalar fields

2.1. Path integral for the partition function

We start by deriving a path integral representation for scalar field theory, by making use of the result obtained for the quantum mechanical harmonic oscillator (HO) in the previous section.

In quantum field theory, the form of the theory is most economically defined in terms of the corresponding classical Lagrangian $\mathcal{L}_M$, rather than the Hamiltonian $\hat{H}$ (for instance, Lorentz symmetry is explicit only in $\mathcal{L}_M$). Let us therefore start from Eq. (1.7), and re-interpret x as an "internal" degree of freedom ϕ, situated at the origin $\mathbf{0}$ of d-dimensional space, like in Eq. (1.11):

$$S_M^{\rm HO} = \int {\rm d}t\, \mathcal{L}_M^{\rm HO} \;, \tag{2.1}$$

$$\mathcal{L}_M^{\rm HO} = \frac{m}{2}\left(\frac{\partial\phi(t,\mathbf{0})}{\partial t}\right)^2 - V(\phi(t,\mathbf{0}))\;. \tag{2.2}$$

Let us compare this with the usual action of scalar field theory (SFT) in d-dimensional space:

$$S_M^{\rm SFT} = \int {\rm d}t \int {\rm d}^d\mathbf{x}\, \mathcal{L}_M^{\rm SFT} \;, \tag{2.3}$$

$$\mathcal{L}_M^{\rm SFT} = \frac{1}{2}\,\partial^\mu\phi\partial_\mu\phi - V(\phi) = \frac{1}{2}\,(\partial_t\phi)^2 - \frac{1}{2}\,(\partial_i\phi)(\partial_i\phi) - V(\phi)\;, \tag{2.4}$$

where we assume that repeated indices are summed over (irrespective of whether they are up and down, or both at the same altitude), and the metric is $(+---)$.

Comparing Eq. (2.2) with Eq. (2.4) we see that formally, scalar field theory is nothing but a collection (sum) of almost independent harmonic oscillators with $m = 1$, one at every $\mathbf{x}$. These oscillators only interact via the derivative term $(\partial_i\phi)(\partial_i\phi)$ which, in the language of statistical physics, couples nearest neighbours:

$$\partial_i\phi \approx \frac{\phi(t,\mathbf{x}+\epsilon\hat{i}) - \phi(t,\mathbf{x})}{\epsilon}\;, \tag{2.5}$$

where $\hat{i}$ is a unit vector in the direction i.

We then realise, however, that such a coupling does not change the derivation of the path integral (for the partition function) in Sec. 1.3 in any essential way: it was only important that the Hamiltonian was quadratic in the *canonical momenta*, $p = m\dot{x} \leftrightarrow \partial_t\phi$. In other words, the derivation of the path integral is only concerned with objects having to do with the time coordinate (or time derivatives), and these appear identically in Eqs. (2.2) and (2.4). Therefore, we can directly take over the result from Eqs. (1.41)–(1.44):

$$\mathcal{Z}^{\rm SFT}(T) = \int_{\phi(\beta\hbar,\mathbf{x})=\phi(0,\mathbf{x})} \prod_{\mathbf{x}} [\, C\,\mathcal{D}\phi(\tau,\mathbf{x})\,] \exp\left[-\frac{1}{\hbar}\int_0^{\beta\hbar}{\rm d}\tau \int {\rm d}^d\mathbf{x}\,\mathcal{L}_E^{\rm SFT}\right]\;, \tag{2.6}$$

$$\mathcal{L}_E^{\rm SFT} = -\mathcal{L}_M^{\rm SFT}(t\to -i\tau) = \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \sum_{i=1}^{d}\frac{1}{2}\left(\frac{\partial\phi}{\partial x^i}\right)^2 + V(\phi)\;. \tag{2.7}$$

We will drop out the superscript SFT in the following and also, for brevity, mostly write $\mathcal{L}_E$ in the form $\mathcal{L}_E = \frac{1}{2}\,\partial_\mu\phi\partial_\mu\phi + V(\phi)$.

2.2. Fourier representation

We now parallel the strategy in Sec. 1.4, and rewrite the path integral in Fourier representation. In order to simplify the notation somewhat, we measure time in units where $\hbar = 1.05 \times 10^{-34}$ Js $= 1$. Then the dependence of τ can be expressed as

$$\phi(\tau, \mathbf{x}) = T \sum_{n=-\infty}^{\infty} \tilde{\phi}(\omega_n, \mathbf{x}) e^{i\omega_n \tau} \; , \quad \omega_n = 2\pi T n \; , n \in \mathbb{Z} \; . \tag{2.8}$$

For the space coordinates, it is useful to momentarily take each direction finite, of extent L_i, and impose periodic boundary conditions, just like for the time direction. Then the dependence on x_i can be represented as

$$f(x_i) = \frac{1}{L_i} \sum_{n_i=-\infty}^{\infty} \tilde{f}(n_i) e^{ik_i x_i} \; , \quad k_i = \frac{2\pi n_i}{L_i} \; , \quad n_i \in \mathbb{Z} \; , \tag{2.9}$$

where $1/L_i$ plays the same role as T in the time direction. In the infinite volume limit, the sum in Eq. (2.9) goes over to the usual Fourier integral,

$$\frac{1}{L_i} \sum_{n_i} = \frac{1}{2\pi} \sum_{n_i} \Delta k_i \stackrel{L_i \to \infty}{\longrightarrow} \int \frac{\mathrm{d}k_i}{2\pi} \; , \tag{2.10}$$

so that the finite volume is really just an intermediate regulator. The whole function in Eq. (2.8) now becomes

$$\phi(\tau, \mathbf{x}) = T \sum_{\omega_n} \frac{1}{V} \sum_{\mathbf{k}} \tilde{\phi}(\omega_n, \mathbf{k}) e^{i\omega_n \tau + i\mathbf{k}\cdot\mathbf{x}} \; , \quad V \equiv L_1 L_2 ... L_d \; . \tag{2.11}$$

Like in Sec. 1.4, the reality of $\phi(\tau, \mathbf{x})$ implies that the Fourier modes satisfy

$$\Big[\tilde{\phi}(\omega_n, \mathbf{k})\Big]^* = \tilde{\phi}(-\omega_n, -\mathbf{k}) \; . \tag{2.12}$$

Thereby only half of the Fourier-modes are independent; we can choose, for instance,

$$\tilde{\phi}(\omega_n, \mathbf{k}) \; , \quad n \geq 1 \; ; \quad \tilde{\phi}(0, \mathbf{k}) \; , \quad k_1 > 0 \; ; \quad \tilde{\phi}(0, 0, k_2, ...) \; , \quad k_2 > 0 \; ; \; \ldots \; ; \quad \text{and} \quad \tilde{\phi}(0, \mathbf{0}) \tag{2.13}$$

as the integration variables. Note again the presence of a zero-mode.

Quadratic forms can be written as

$$\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x} \, \phi_1(\tau, \mathbf{x}) \phi_2(\tau, \mathbf{x}) = T \sum_{\omega_n} \frac{1}{V} \sum_{\mathbf{k}} \tilde{\phi}_1(-\omega_n, -\mathbf{k}) \tilde{\phi}_2(\omega_n, \mathbf{k}) \; . \tag{2.14}$$

In particular, in the *free case*, i.e. for $V(\phi) \equiv \frac{1}{2} m^2 \phi^2$, the exponent in Eq. (2.6) becomes

$$\begin{aligned} \exp(-S_E) &= \exp\Big(-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d\mathbf{x} \, \mathcal{L}_E\Big) \\ &= \exp\Big[-\frac{1}{2} T \sum_{\omega_n} \frac{1}{V} \sum_{\mathbf{k}} (\omega_n^2 + \mathbf{k}^2 + m^2) |\tilde{\phi}(\omega_n, \mathbf{k})|^2\Big] \\ &= \prod_{\mathbf{k}} \Big\{ \exp\Big[-\frac{T}{2V} \sum_{\omega_n} (\omega_n^2 + \mathbf{k}^2 + m^2) |\tilde{\phi}(\omega_n, \mathbf{k})|^2\Big]\Big\} \; . \end{aligned} \tag{2.15}$$

The exponential here is precisely the same as the one in Eq. (1.75), with the replacements

$$m^{(\mathrm{HO})} \to \frac{1}{V}\;, \quad \omega^{2(\mathrm{HO})} \to \mathbf{k}^2 + m^2\;, \quad |x_n^2|^{(\mathrm{HO})} \to |\tilde{\phi}(\omega_n, \mathbf{k})|^2\;. \tag{2.16}$$

The result thus factorises into a *product of harmonic oscillator partition functions*, for which we know the answer already. In fact, rewriting Eqs. (1.54), (1.59), (1.18) for the case $\hbar = 1$, the harmonic oscillator partition function can be represented as

$$\mathcal{Z}^{\mathrm{HO}} = C' \int \left[\prod_{n\geq 0} \mathrm{d}x_n\right] \exp\left[-\frac{mT}{2} \sum_{n=-\infty}^{\infty} (\omega_n^2 + \omega^2)|x_n|^2\right] \tag{2.17}$$

$$= \frac{T}{\omega} \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega^2 + \omega_n^2} \tag{2.18}$$

$$= T \prod_{n=-\infty}^{\infty} (\omega_n^2 + \omega^2)^{-\frac{1}{2}} \prod_{n'=-\infty}^{\infty} (\omega_n^2)^{\frac{1}{2}} \tag{2.19}$$

$$= \exp\left\{-\frac{1}{T}\left[\frac{\omega}{2} + T \ln\left(1 - e^{-\beta\omega}\right)\right]\right\}\;, \tag{2.20}$$

where n' means that the zero-mode $n = 0$ is omitted. Note in particular that all dependence on $m^{(\mathrm{HO})}$ has dropped out.

Combining Eq. (2.15) with Eqs. (2.17)–(2.20), we obtain two useful representations for $\mathcal{Z}^{\mathrm{SFT}}$. First of all, denoting

$$E_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + m^2}\;, \tag{2.21}$$

Eq. (2.19) yields

$$\mathcal{Z}^{\mathrm{SFT}} = \exp\left(-\frac{F^{\mathrm{SFT}}}{T}\right) = \prod_{\mathbf{k}} \left\{T \prod_n (\omega_n^2 + E_{\mathbf{k}}^2)^{-\frac{1}{2}} \prod_{n'} (\omega_n^2)^{\frac{1}{2}}\right\} \tag{2.22}$$

$$= \exp\left\{\sum_{\mathbf{k}} \left[\ln T + \frac{1}{2}\sum_{n'} \ln \omega_n^2 - \frac{1}{2}\sum_n \ln(\omega_n^2 + E_{\mathbf{k}}^2)\right]\right\}. \tag{2.23}$$

Taking then the infinite-volume limit, the free-energy density, F/V, can be written as

$$\lim_{V\to\infty} \frac{F^{\mathrm{SFT}}}{V} = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \left[T \sum_{\omega_n} \frac{1}{2} \ln(\omega_n^2 + E_{\mathbf{k}}^2) - T \sum_{\omega_n'} \frac{1}{2} \ln(\omega_n^2) - T\frac{1}{2} \ln(T^2)\right]. \tag{2.24}$$

Second, making directly use of Eq. (2.20), we get

$$\mathcal{Z}^{\mathrm{SFT}} = \exp\left(-\frac{F^{\mathrm{SFT}}}{T}\right) = \prod_{\mathbf{k}} \left\{\exp\left[-\frac{1}{T}\left(\frac{E_{\mathbf{k}}}{2} + T\ln\left(1 - e^{-\beta E_{\mathbf{k}}}\right)\right)\right]\right\}\;, \tag{2.25}$$

$$\lim_{V\to\infty} \frac{F^{\mathrm{SFT}}}{V} = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \left[\frac{E_{\mathbf{k}}}{2} + T \ln\left(1 - e^{-\beta E_{\mathbf{k}}}\right)\right]. \tag{2.26}$$

We will return to the evaluation of the momentum integration in Secs. 2.5 and 2.6.