Basics of thermal QCD

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Abstract:

It has been a longstanding dream that experimental tests of thermal QCD through heavy ion collision experiments could yield theoretical insights that would be useful for some cosmological problems as well. These lectures cover selected topics within thermal QCD with this perspective in mind. The observables touched upon are the equation of state, viscosities, as well as the rates of elastic and inelastic reactions experienced by heavy quarks. Depending on the observable the focus will be either on elaborating on the basic concepts, on outlining the link between heavy ion collisions and cosmology, or on reviewing modern developments.

Contents:

Further material:

`www.physik.uni-bielefeld.de/~laine/thermal/`

I. Overall equation of state

QCD

Continuum theory in Euclidean signature:

$$\mathcal{L}_{\mathsf{QCD}} = \frac{1}{4g^2}\sum_{a=1}^{N_{\rm c}^2-1} F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{i=1}^{N_{\rm f}} \bar\psi_i[\gamma_\mu D_\mu + m_i]\psi_i \;.$$

$N_{\rm c} = N_{\rm f} = 3$; in the $\overline{\rm MS}$ renormalization scheme with a scale ~ 2 GeV, $m_u, m_d \sim 5$ MeV, $m_s \sim 100$ MeV; $g(M_Z) \sim 1.2$.

We regulate here mostly by going into $D = 4 - 2\epsilon$ dimensions.

Thermodynamics

Minus grand canonical free energy density, i.e. pressure:

$$p(T,\mu) \equiv \lim_{V\to\infty} \frac{T}{V} \ln \left\{ \mathrm{Tr} \left[\exp\left(-\frac{\hat{H}_{\text{QCD}} - \mu \hat{B}}{T} \right) \right] \right\},$$

where $\hat{H}_{\text{QCD}}$ is the Hamilton operator corresponding to $\mathcal{L}_{\text{QCD}}$, and $\hat{B}$ is the baryon number operator.

We will denote $p(T) \equiv p(T,0)$, which is the "hot" case. One can also consider the "dense" case $\mu \neq 0$, relevant for astrophysics; however in cosmological applications $|\mu| \ll T$.

In cosmology, according to the Einstein equations, the cooling rate of the Universe is

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\sqrt{24\pi}}{m_{\text{Pl}}}\frac{\sqrt{e(T)}s(T)}{c(T)} \, ,$$

where $m_{\text{Pl}} = 1.2 \times 10^{19}$ GeV and

$$\begin{aligned} s(T) &\equiv p'(T) \; \text{[entropy density]}, \\ e(T) &\equiv Ts(T) - p(T) \; \text{[energy density]}, \\ c(T) &\equiv e'(T) = Tp''(T) \; \text{[heat capacity]}. \end{aligned}$$

Cosmological **relics** (dark matter, background radiation, etc) are born when some "microscopic" reaction time $\tau(T)$ becomes longer than the "macroscopic" time period $t_{\text{now}} - t(T)$.

For instance, for "**Cold Dark Matter**" of mass M, experiencing only weak interactions, this happens at $T \sim M/25$. For $M =$ 10...1000 GeV, $T =$ 0.4...40 GeV, and QCD is important.

Srednicki Watkins Olive NPB 310 (1988) 693;
Hindmarsh Philipsen hep-ph/0501232

For "**Warm Dark Matter**" made of right-handed "sterile" neutrinos with $M \sim$ keV, production peaks at $T \sim 150$ MeV. Then QCD effects are even more important.

Dodelson Widrow hep-ph/9303287;
Shi Fuller Abazajian astro-ph/9810076, astro-ph/0204293
Asaka ML Shaposhnikov hep-ph/0612182; ML Shaposhnikov 0804.4543

We will return to specific examples later on!

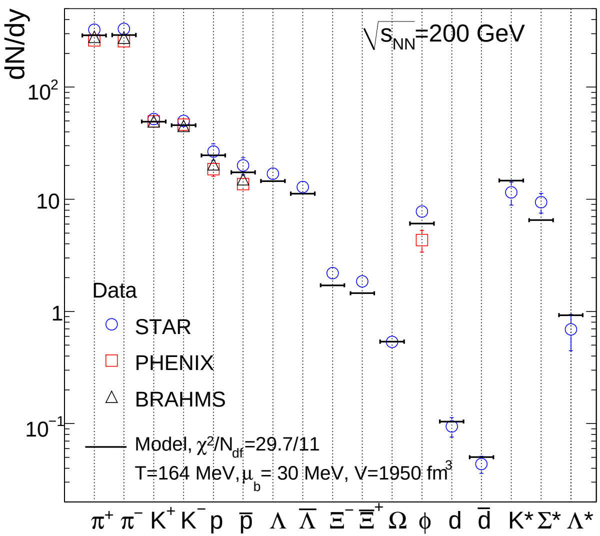
In heavy ion collision experiments, the expansion of the system, after thermalization, is determined by the energy-momentum tensor

$$T^{\mu\nu} = [p(T) + e(T)]u^\mu u^\nu - p(T)g^{\mu\nu} + \mathcal{O}(\partial^\mu u^\nu) \ ,$$

where u^μ is the flow velocity, and $\partial_\mu T^{\mu\nu} = 0$.

After hydrodynamic expansion the system gets hadronized at $T \sim 100...150$ MeV. The hadron spectrum observed depends indirectly on $p(T)$.

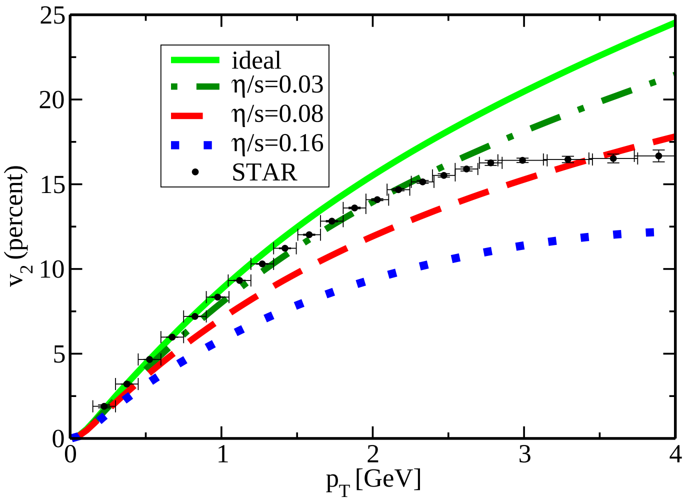
Actually, at first sight, it is not easy to extract $p(T)$. Total multiplicity in heavy ion collisions:



Andronic et al 0901.2909

All information about evolution before hadronization appears to be lost: thermal equilibrium has no memory!

More sensitive are differential observables like "elliptic flow", v_2, characterizing the anisotropy of momentum distribution in various directions of a non-central collision:



2×Romatschke 0706.1522

In this example **none** of the models describes the data, so assumed $p(T)$ or other inputs would need to be modified!

Solid understanding of $p(T)$ is possible in 2 limits only.

0 MeV $\lesssim T \lesssim 100$ MeV: chiral symmetry breaking + confinement ⇒ weakly interacting massive hadrons:

$$p(T) \approx \sum_i T^4 \left(\frac{m_i}{2\pi T}\right)^{\frac{3}{2}} e^{-\frac{m_i}{T}} \; .$$

$T \gg 500$ MeV: asymptotic freedom ⇒ weakly interacting quarks and gluons:

$$p(T) \approx \frac{\pi^2 T^4}{90} \left[2(N_\mathrm{c}^2 - 1) + \frac{7}{2} N_\mathrm{c} N_\mathrm{f}\right] \approx 5.2 T^4 \; .$$

What happens at intermediate temperatures?

Traditionally it was thought that there is a strong first order phase transition in between the low-T and the high-T regimes.

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Cosmic separation of phases

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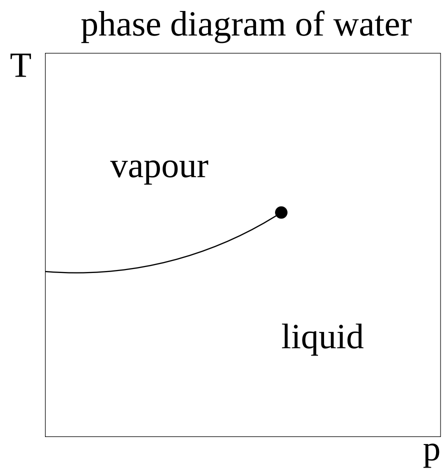
(Received 9 April 1984)

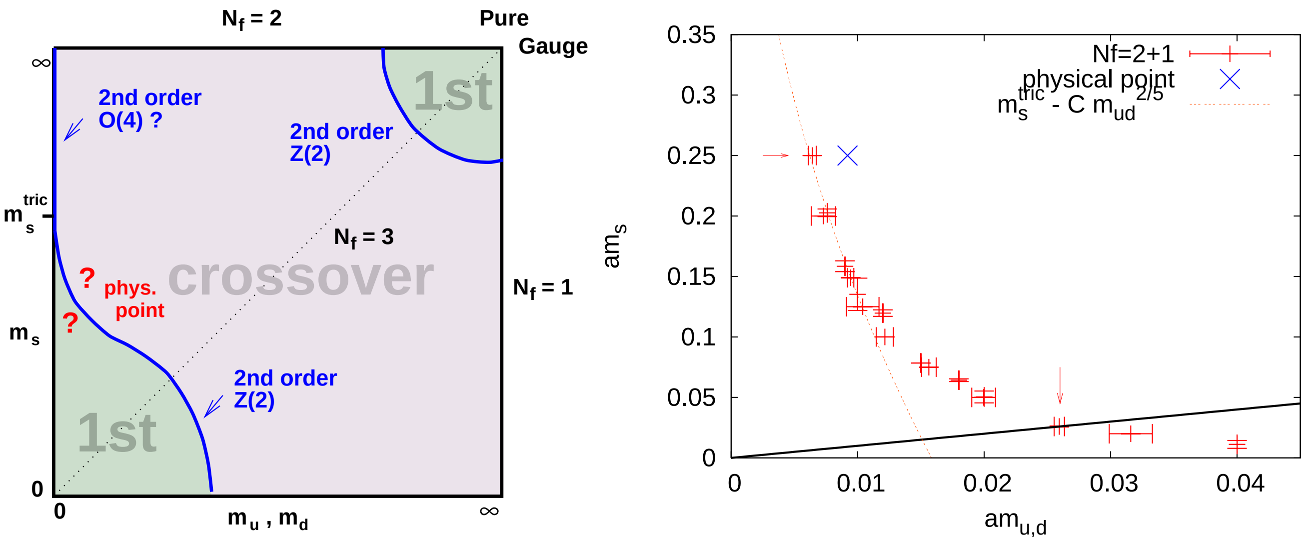
A first-order QCD phase transition that occurred reversibly in the early universe would lead to a surprisingly rich cosmological scenario. Although observable consequences would not necessarily survive, it is at least conceivable that the phase transition would concentrate most of the quark excess in dense, invisible quark nuggets, providing an explanation for the dark matter in terms of QCD effects only. This possibility is viable only if quark matter has energy per baryon less than 938 MeV. Two related issues are considered in appendices: the possibility that neutron stars generate a quark-matter component of cosmic rays, and the possibility that the QCD phase transition may have produced a detectable gravitational signal.

But the distinction between the two limits could also be smooth, like in between liquid and vapour, because QCD has **no "order parameter"** for physical quark masses.



("Spontaneous chiral symmetry breaking" and "confinement" can be precisely defined only in specific limits.)

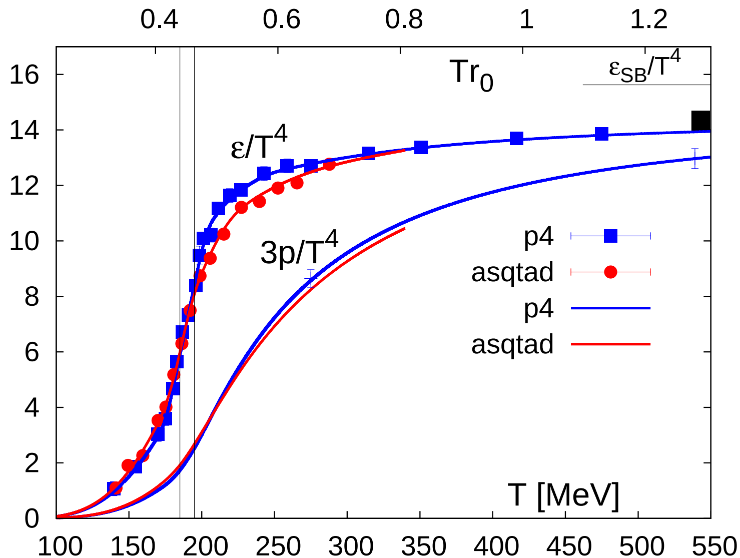
To find out what really happens, lattice experiments have been carried out since more than 25 years.



de Forcrand Philipsen hep-lat/0607017; Aoki et al hep-lat/0611014

However systematic errors (volume, lattice spacing, non-physical breaking of chiral symmetry) are hard to quantify.

Pressure as a function of T in the crossover region:



hotQCD 0903.4379; Aoki et al 0903.4155; Kanaya et al 0910.5284; Bornyakov et al 0910.2392

Systematic errors possible here as well, but are gradually being reduced through the work of many groups.

Unfortunately numerical results can yield no analytic control over parameteric dependences $(N_c, N_{\rm f}, m_u, m_d, m_s)$, which would be interesting from the **theoretical** point of view.

$N_{\rm c}$-dependence at $N_{\rm f} = 0$: Datta Gupta 0910.2889; Panero 0907.3719

They are also hard to extrapolate to very high temperatures (1 GeV $\lesssim T \lesssim$ 100 GeV), interesting from the **cosmological** point of view, if "Cold Dark Matter" is realized in nature.

So, it is worthwhile to explore complementary avenues as well!

pQCD at high temperatures

Lattice (p.16) deviates from non-interacting quarks and gluons ($\equiv \epsilon_{\mathsf{SB}}(T)$; SB for "Stefan-Boltzmann") even at $T \sim 550$ MeV. Could the deviation be understood as a "small correction"?

To find out, compute corrections to $p_{\mathsf{SB}}(T)$ in a power series in the QCD coupling constant g!

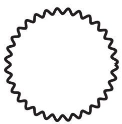
g^2:	Shuryak 1978; Chin 1978
g^3:	Kapusta 1979
$g^4 \ln(1/g)$:	Toimela 1983
g^4:	Arnold, Zhai 1994
g^5:	Zhai, Kastening 1995; Braaten, Nieto 1995
$g^6 \ln(1/g)$:	Schröder 2002; Kajantie et al 2002
g^6 (partly):	Di Renzo et al 2006; Gynther et al 2009

How does it go?

Start from a **path integral representation** of $p(T)$:

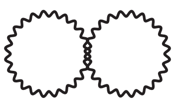
$$p(T) = \lim_{V\to\infty} \frac{T}{V} \ln \int_{\text{b.c.}} \mathcal{D}[A^a_\mu, \bar\psi, \psi] e^{-\int_0^{1/T} d\tau \int d^{3-2\epsilon}\mathbf{x}\, \mathcal{L}_{\text{QCD}}} \, .$$

1-loop:



$$p(T) = \frac{\pi^2 T^4}{45}(N_c^2 - 1 + \frac{7}{4} N_c N_f) \, .$$

2-loop:

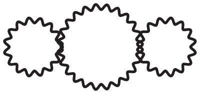


$$\delta p(T) = -\frac{g^2 T^4}{144}(N_c^2 - 1)(N_c + \frac{5}{4} N_f) \, .$$

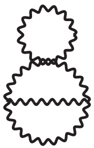
Example of a sum-integral contributing to $\delta p(T)$:

$$
\begin{aligned}
& T \sum_{\omega_n} \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{1}{\omega_n^2 + \mathbf{k}^2} \\
= \quad & 2T \sum_{n=1}^{\infty} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)} \frac{1}{(2\pi n T)^{2-d}} \\
= \quad & 2T \frac{1}{(4\pi)^{d/2}(2\pi T)^{2-d}} \Gamma(1-\frac{d}{2}) \zeta(2-d) \\
\stackrel{d=3-2\epsilon}{=} \quad & \mu^{-2\epsilon} \frac{T^2}{12} \Big\{ 1 \qquad \Big\| \quad {\color{red} \zeta(-1) = -\frac{1}{12}!} \\
& + \epsilon \left[2 \ln \left(\frac{\bar{\mu} e^{\gamma_{\mathrm{E}}}}{4\pi T} \right) + 2 - 2\gamma_{\mathrm{E}} + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] \\
& + \mathcal{O}(\epsilon^2) \Big\} , \quad \bar{\mu}^2 \equiv 4\pi \mu^2 e^{-\gamma_{\mathrm{E}}} .
\end{aligned}
$$

3-loop: Uncancelled infrared (IR) divergences!



$$g^4 T^5 \int \frac{d^3k}{k^4} + \ldots$$ power divergent .



$$g^4 T^4 \int \frac{d^3k \, d^3q}{k^2 q^2 (k+q)^2} + \ldots$$ log divergent .

Strict perturbation theory (loop expansion) breaks down!

Loop expansion $\neq$ weak-coupling expansion

Physics: interactions make it a **multiscale** system, generating "collective phenomena" like colour-electric screening at $|\mathbf{k}| \sim m_{\mathsf{E}} \sim gT$, and colour-magnetic screening at $|\mathbf{k}| \sim g^2T/\pi$.

We did not account for this, thus met a dead end.

Method for making progress: **effective field theories**.

QCD; $|\mathbf{k}| \sim \pi T, gT, g^2T/\pi$

$\Downarrow$ perturbation theory (1)

EQCD; $|\mathbf{k}| \sim gT, g^2T/\pi$

$\Downarrow$ perturbation theory (2)

MQCD; $|\mathbf{k}| \sim g^2T/\pi$

$\Downarrow$ numerical simulations (3)

$p(T)$.

$$\left(\quad \mathcal{L}_{\mathsf{EQCD}} = \frac{1}{4g_{\mathrm{E}}^2} F_{ij}^a F_{ij}^a + \frac{1}{2} (\mathcal{D}_i^{ab} A_0^b)(\mathcal{D}_i^{ac} A_0^c) + m_{\mathsf{E}}^2 \, \mathrm{Tr}[A_0^2] + \ldots \, . \quad \right)$$

Contributions: ($\Lambda_{\mathsf{E}}, \Lambda_{\mathsf{M}}$ are "matching scales", i.e. $\bar{\mu}$'s associated with the various steps; for step (1) p.20 with $\bar{\mu} \longrightarrow \Lambda_{\mathsf{E}}$)

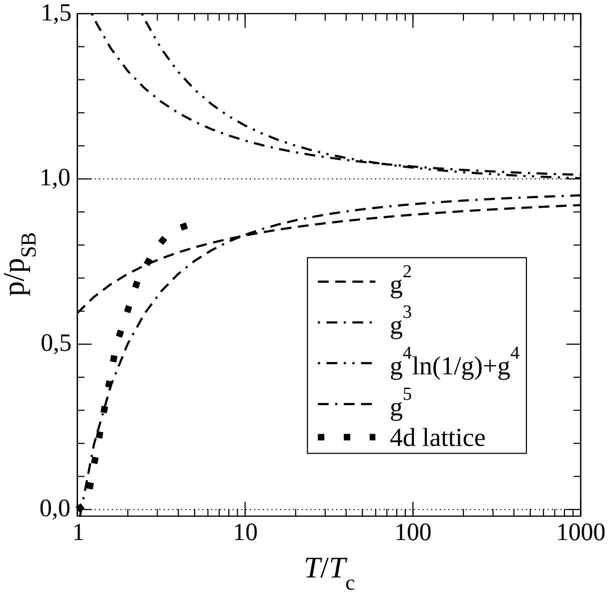
$$\frac{\delta p_{(1)}}{T^4} \sim 1 + g^2 + g^4 \ln \frac{4\pi T}{\Lambda_{\mathsf{E}}} + g^6 \ln \frac{4\pi T}{\Lambda_{\mathsf{E}}} + \ldots ,$$

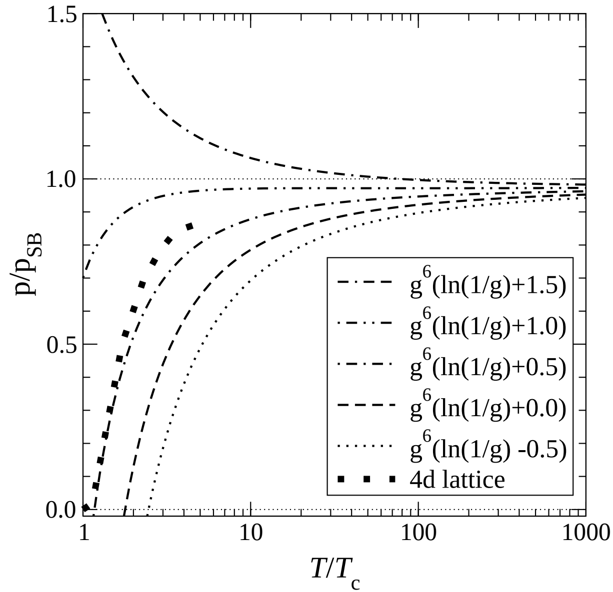
$$\frac{\delta p_{(2)}}{T^4} \sim g^3 + g^4 \ln \frac{\Lambda_{\mathsf{E}}}{gT} + g^5 + g^6 \ln \frac{\Lambda_{\mathsf{E}}}{gT} + g^6 \ln \frac{gT}{\Lambda_{\mathsf{M}}} + \ldots ,$$

$$\frac{\delta p_{(3)}}{T^4} \sim g^6 \left(\ln \frac{\Lambda_{\mathsf{M}}}{g^2 T/\pi} + [\text{non-pert}] \right) .$$

If the effective theories are the correct ones, then the unphysical matching scales cancel at the end, which serves as a nice consistency check for the computation.

Numerical evaluation ($N_\mathrm{f} = 0$, $T_\mathrm{c} \simeq 0.3$ GeV):

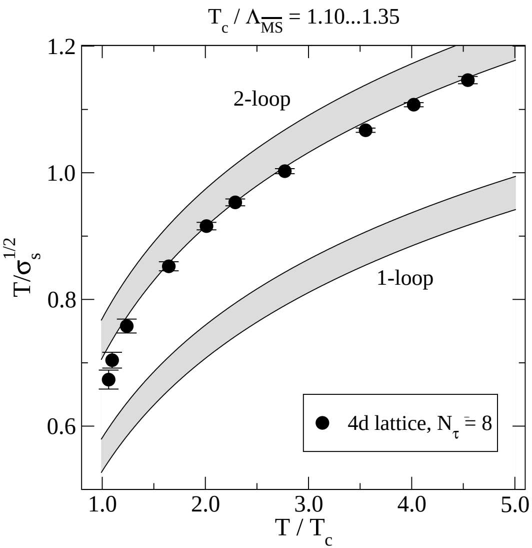


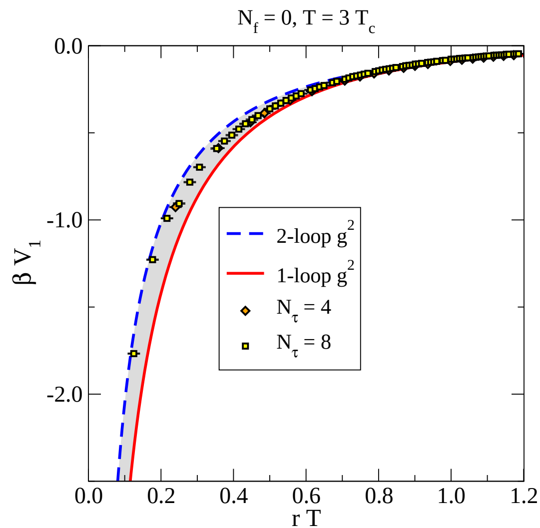


Kajantie et al hep-ph/0211321

$\Rightarrow$ Interactions are strong even at high T — system is **not free**.
[Reason for big effects (?): first contribution from a new scale.]

Nevertheless there is hope that interactions are accountable: several past tests have worked out nicely!





Spatial string tension

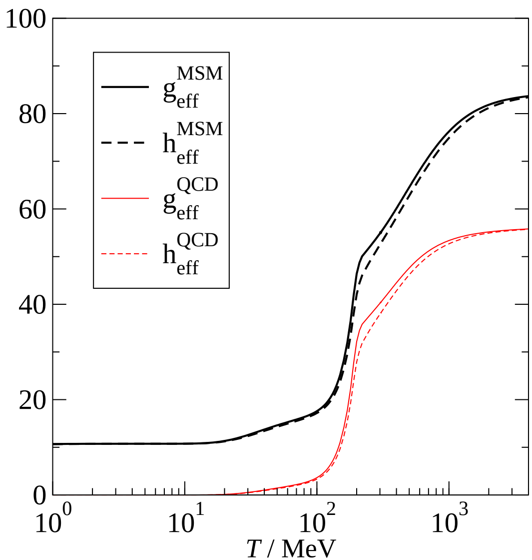
Heavy quark potential

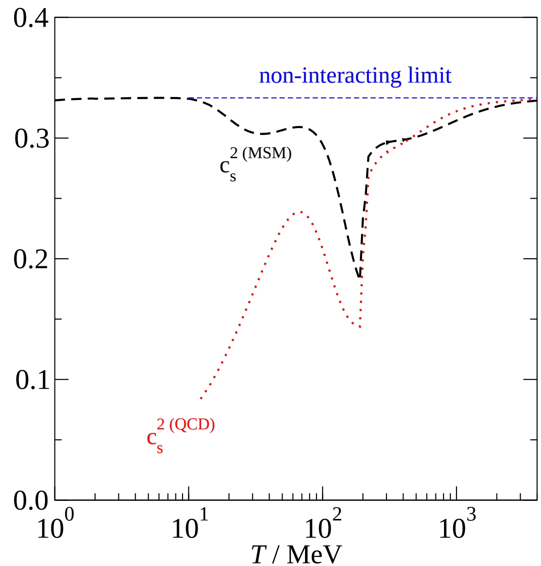
ML Schröder hep-ph/0503061

Burnier et al 0911.3480

Subsequently $N_\mathrm{f} \to 6$ and include weakly interacting particles. E.g. interpolating resonance gas at $T < T_\mathrm{c}$ to pQCD at $T > T_\mathrm{c}$:

$$g_\mathrm{eff} \equiv \frac{e(T)}{[\frac{\pi^2 T^4}{30}]} , \quad h_\mathrm{eff} \equiv \frac{s(T)}{[\frac{2\pi^2 T^3}{45}]} , \quad c_s^2 \equiv \frac{\mathrm{d}p}{\mathrm{d}e} .$$





ML Schröder hep-ph/0603048

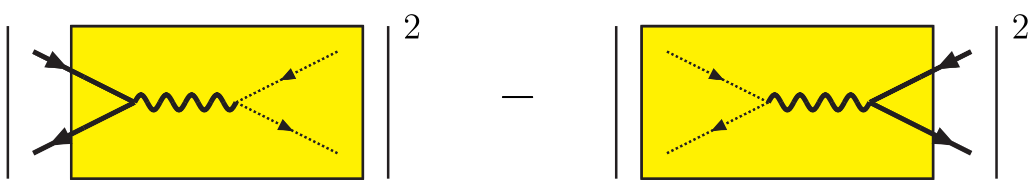
With such curves we could return to cosmology and consider the evolution equation for the Cold Dark Matter relic abundance.

For $Y(T) \equiv n_{\rm CDM}(T)/s(T)$:   Gondolo Gelmini NPB 360 (1991) 145

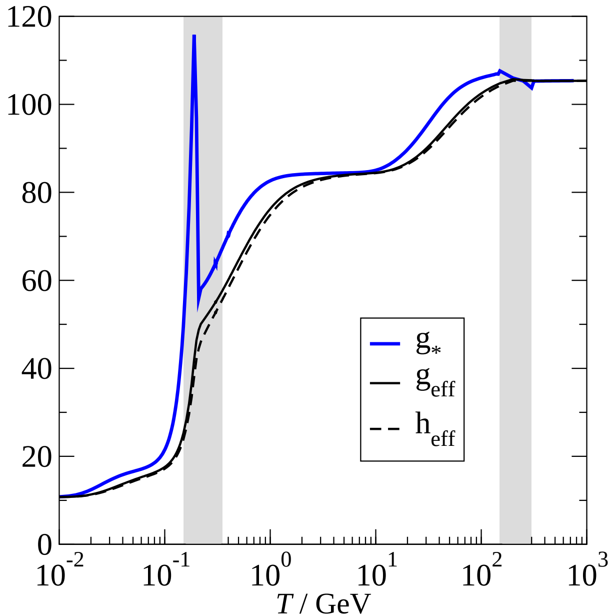
$$\frac{{\rm d}Y}{{\rm d}T} \simeq \sqrt{\frac{\pi g_*(T)}{45}}\, m_{\rm Pl} \, \langle \sigma v_{\rm M\o l} \rangle \, (Y^2 - Y_{\rm eq}^2) \ ,$$

where

$$g_*(T) \simeq \frac{h_{\rm eff}^2(T)}{g_{\rm eff}(T)} \left[\frac{1}{3 c_s^2(T)} \right]^2 \ .$$



Inserting previous functions we can plot g_*:



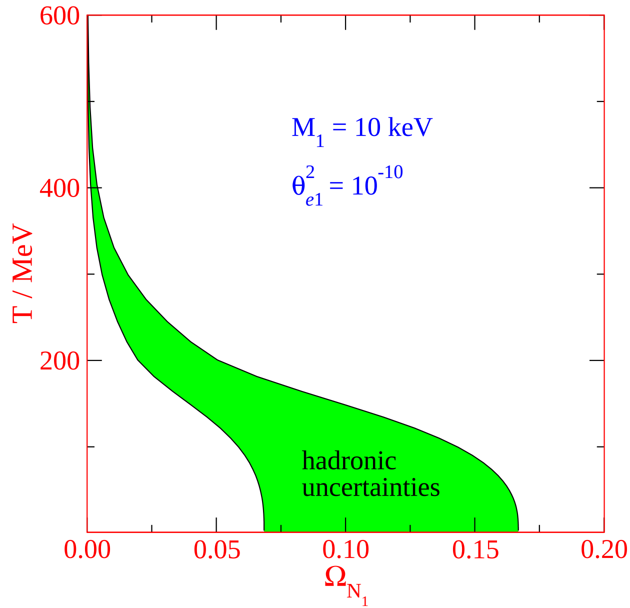
Speed of sound p'/Tp'' has a dip $\Rightarrow$ heat capacity Tp'' has a peak $\Rightarrow$ it takes extra time to dilute all the heat released.

In the grey zones lattice data on the QCD and electroweak phase transitions, respectively, would be needed for precise statements.

Effects of uncertainties on Y on the 1% level.

For Warm Dark Matter, production can happen close to the QCD crossover, leading to more dramatic effects ($\Omega \propto M_{\rm WDM} Y$):

Asaka ML Shaposhnikov hep-ph/0612182



Effects of uncertainties on the 10% – 100% level.

Conclusions

From the theoretical as well as phenomenological point of view, one of the important observables of QCD is minus the free energy density, or pressure, $p(T)$.

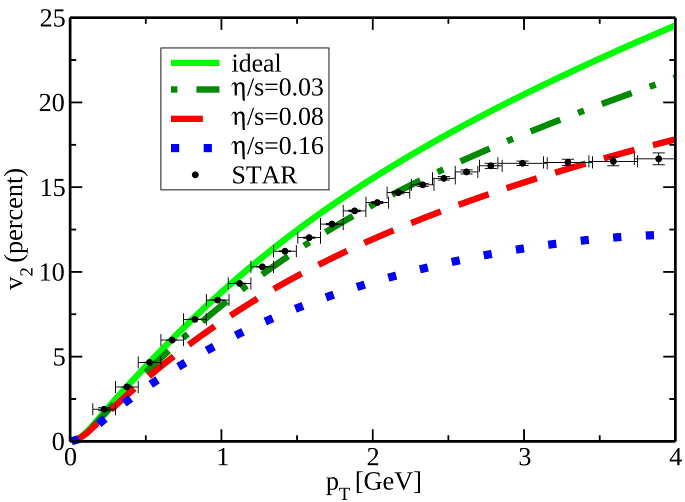
Its computation is plagued by IR sensitivity, even at very high temperatures $T \gg$ GeV (or very small couplings $g \ll 1$).

Effective theory methods allow to resum appropriate classes of higher loop diagrams, either perturbatively (effects from the scale $m_{\mathsf{E}} \sim gT$) or non-perturbatively (effects from the scale g^2T/π), so that a weak-coupling expansion can be defined.

The results have significance for certain Dark Matter scenarios.

II. Shear and bulk viscosities

Recall again that in principle the hydrodynamic characteristics of the QCD plasma can be extracted from a comparison between heavy ion collision experiments and simulation:



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Here certain $p(T)$, $e(T)$, $s(T)$ were assumed as fixed inputs, and $\eta(T)$, the so-called "shear viscosity", was varied.

The reason that a viscosity plays a role is that it yields a **gradient correction** to the energy-momentum tensor:

$$T^{\mu\nu} = [p(T) + e(T)]u^\mu u^\nu - p(T)g^{\mu\nu} + \mathcal{O}((\eta, \zeta)\partial^\mu u^\nu) \ .$$

Shear viscosity η: traceless part. Bulk viscosity ζ: trace part. Explicitly, in the non-relativistic limit $|u^\mu| \ll 1$:

$$T_{ij} = (p - \zeta \nabla \cdot \mathbf{v})\delta_{ij} - \eta(\partial_i v^j + \partial_j v^i - \frac{2}{3}\delta_{ij}\nabla \cdot \mathbf{v}) \ .$$

Since the system generated in heavy ion collisions is **small**, gradients can be large: $\eta\partial u \sim p$.

In contrast, the Universe is very homogeneous, and gradient corrections can have no direct effect on the overall expansion.[1]

For instance, in the QCD epoch ($T \sim 200$ MeV), the microscopic length scale is $\xi \sim \frac{1}{T}$ while the system size = horizon radius is

$$\ell_H \sim t_{\rm Universe} \sim \frac{m_{\sf Pl}}{T^2} \sim \frac{10^{20}}{T} \, .$$

It turns out, however, that the bulk viscosity ζ makes a formal appearance in a completely different context!

Bödeker hep-ph/0605030

[1] Evolution of density perturbations may be a different story.

Consider some very weakly coupled scalar field, φ:

$$\mathcal{L}_\varphi \sim \frac{1}{2}\varphi(-\Box - m^2)\varphi + \frac{\varphi}{M} \times \frac{1}{g^2} F^{a\mu\nu} F^a_{\mu\nu} ,$$

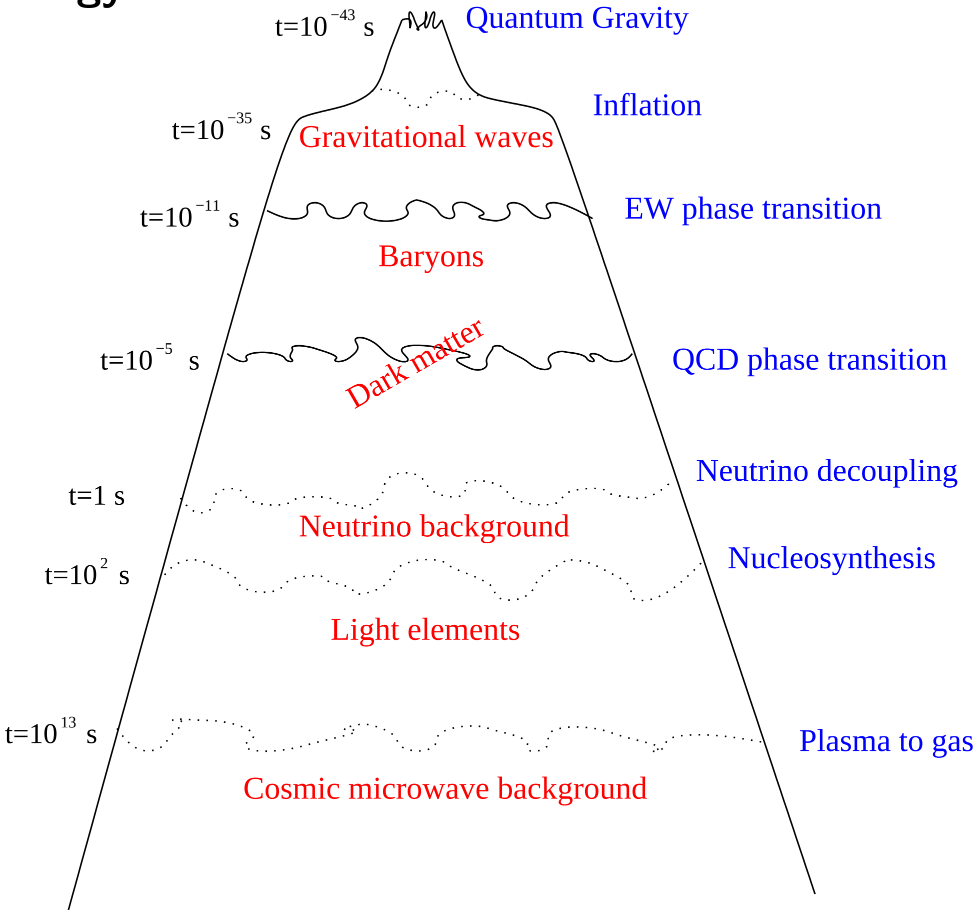
where $m \sim m_{\mathrm{SUSY}}$, $M \sim m_{\mathrm{Pl}}$.

This leads to a "moduli problem": after inflation $\langle \varphi(0) \rangle \gg m$, and φ decays slowly, $\Gamma \sim m^3/M^2$, whereby its energy density eventually comes to dominate over that of radiation.

This is in contrast with standard cosmology, where a radiation-dominated expansion is needed at least starting from the nucleosynthesis epoch.

Standard cosmology

$$\frac{t}{\mathrm{s}} \approx \left(\frac{\mathrm{MeV}}{T}\right)^2$$



Q: could the vacuum rate $\Gamma \sim m^3/M^2$ be made faster by thermal corrections associated with the "normal" degrees of freedom, represented by $F^a_{\mu\nu}$?

If so, the "dangerous relic" φ could decay (i.e. lose its energy to radiation) before nucleosynthesis, and become harmless.

(There are many other similar dangerous relics in cosmology; in particular theories predicting the generation of topological defects in the form of domain walls or monopoles are practically excluded, if the associated energy scale corresponds to physics beyond the Standard Model.)

Equation of motion in a non-trivial environment:

$$\Box\varphi + V'_{\rm eff}(\varphi) = -\Gamma\dot{\varphi} + \mathcal{O}(\dot{\varphi}^2, (\nabla\varphi)^2) \; .$$

The assumption is that φ varies slowly in time and space, so that the "fast" thermal matter fields see it as a constant.

A standard tool in thermal field theory is a "Kubo formula", allowing to determine such "response" coefficients:

$$\Gamma = -\lim_{\omega\to 0}\frac{\rho(\omega,\mathbf{0})}{\omega} \; ,$$

where for $\mathcal{L}_{\rm int} = \varphi H_{\rm int}$:

$$\rho(\omega,\mathbf{0}) = -\int_{t,\mathbf{x}} e^{i\omega t}\left\langle \frac{1}{2}\Big[\hat{H}_{\rm int}(t,\mathbf{x}), \hat{H}_{\rm int}(0,\mathbf{0})\Big]\right\rangle \; .$$

Sketch of a derivation

In Fourier space, with $\varphi \propto e^{-i\omega t + i\mathbf{p}\cdot\mathbf{x}}\,\tilde{\varphi}$:

$$[-\omega^2 + \mathbf{p}^2 + m_{\text{eff}}^2 - i\omega\Gamma]\tilde{\varphi} = \mathcal{O}(\tilde{\varphi}^2)\ .$$

Compare this with the Euclidean propagator of $\tilde{\varphi}$ after analytic continuation $\omega_n \to -i(\omega + i0^+)$:

$$\frac{1}{\omega_n^2 + \mathbf{p}^2 + \Pi_E} \to \frac{1}{-\omega^2 + \mathbf{p}^2 + \operatorname{Re}\Pi_E + i\operatorname{Im}\Pi_E}\ .$$

Take $\mathbf{p} \to \mathbf{0}$; denote $m_{\text{eff}}^2 \equiv \operatorname{Re}\Pi_E$; and $\rho(\omega) \equiv \operatorname{Im}\Pi_E$.

Comparing the "pole position" with the solution of the equation of motion we can identify $\Gamma = -\rho(\omega)/\omega$.

The relevant oscillation frequency would now be

$$\omega^2 = m_{\text{eff}}^2 \sim m^2 + \frac{T^4}{M^2} ,$$

but for $m \ll T \ll M$ this is much smaller than any other scales in the system, particularly T, so we can just as well take the limit $\omega = m_{\text{eff}} \to 0$.

The final step, the rewriting of $\rho = \text{Im}\,\Pi_E$ as a commutator, is a standard relation and will be reviewed in the next lecture.

Let us now return to the specific case $H_{\text{int}} = \frac{1}{M} \times \frac{1}{g^2} F^{a\mu\nu} F^a_{\mu\nu}$.

At this point it is good to realize that the structure appearing can be recognized as the trace anomaly of pure Yang-Mills theory,

$$\Theta = T^{\mu}{}_{\mu} \sim \frac{\beta_{g^2}}{g^4} F^{a\mu\nu} F^a_{\mu\nu} \; ,$$

where β_{g^2} is the β-function related to the running coupling.

Sketch of a (Euclidean) “derivation”

With the convention

$$S_E = \int \mathrm{d}\tau \int \mathrm{d}^{3-2\epsilon}\mathbf{x} \left\{ \frac{1}{4g_{\mathrm{B}}^2} F^a_{\mu\nu} F^a_{\mu\nu} \right\}$$

the Euclidean energy-momentum tensor is

$$T_{\mu\nu} = \frac{1}{g_{\mathrm{B}}^2} \left(\frac{1}{4} \delta_{\mu\nu} F^a_{\alpha\beta} F^a_{\alpha\beta} - F^a_{\alpha\mu} F^a_{\alpha\nu} \right) \ .$$

In $\delta_{\mu\mu} = 4 - 2\epsilon$ dimensions its trace is

$$T_{\mu\mu} = -\frac{2\epsilon}{4} \frac{1}{g_{\mathrm{B}}^2} F^a_{\alpha\beta} F^a_{\alpha\beta} \ .$$

The bare coupling is

$$g_{\mathrm{B}}^2 = g^2 - \frac{11 N_{\mathrm{c}}}{3\epsilon} \frac{g^4}{(4\pi)^2} + \mathcal{O}(g^6) \ ,$$

$$\frac{1}{g_{\mathrm{B}}^2} = \frac{1}{g^2} + \frac{11 N_{\mathrm{c}}}{3\epsilon} \frac{1}{(4\pi)^2} + \mathcal{O}(g^2) \ .$$

So,

$$T_{\mu\mu} \simeq -\frac{11 N_{\mathrm{c}}}{6} \frac{1}{(4\pi)^2} F^a_{\alpha\beta} F^a_{\alpha\beta}$$

Kubo relation for the bulk viscosity:

$$\zeta = \frac{1}{9} \lim_{\omega \to 0} \frac{1}{\omega} \int_{t,\mathbf{x}} e^{i\omega t} \left\langle \frac{1}{2} \left[\hat{\Theta}(t, \mathbf{x}), \hat{\Theta}(0, \mathbf{0}) \right] \right\rangle \ .$$

Moreover, motivated by heavy ion collisions, the weak-coupling expression for ζ has been worked out,

$$\zeta \sim \frac{\alpha_s^2 T^3}{\ln(1/\alpha_s)} \ , \quad \alpha_s \equiv \frac{g^2}{4\pi} \ .$$

Arnold Dogan Moore hep-ph/0608012

We see now that $H_{\rm int} = \frac{1}{Mg^2} F^{a\mu\nu} F^a_{\mu\nu} \sim \frac{1}{Mg^2}\Theta$.

In conclusion, the vacuum decay rate, $\Gamma \sim \frac{m^3}{M^2}$, is overtaken at $T \gg m$ by a thermal correction:

$$\begin{aligned}\Gamma &= \lim_{\omega\to 0} \frac{1}{\omega} {\rm Im} \int_{t,\mathbf{x}} e^{i\omega t} \left\langle \frac{1}{2} \left[\hat{H}_{\rm int}(t,\mathbf{x}), \hat{H}_{\rm int}(0,\mathbf{0}) \right] \right\rangle \\ &\sim \frac{\zeta}{M^2 g^4} \sim \frac{1}{\ln(1/\alpha_s)} \frac{T^3}{M^2} \gg \frac{m^3}{M^2} \, .\end{aligned}$$

In other words, the fact that there is already a plasma present “facilitates” the decays into normal matter fields.

Conclusions

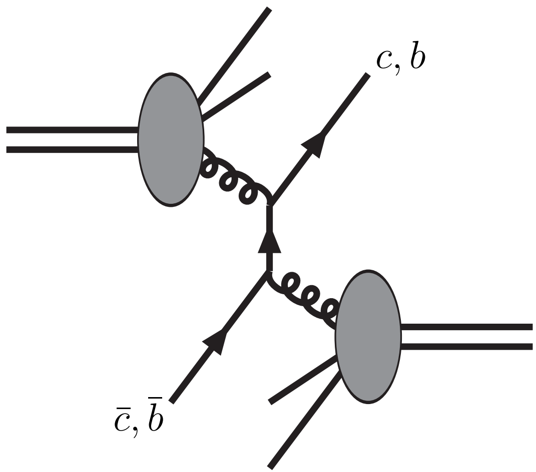
Systematic heavy ion collision inspired computations may find “exciting” applications in totally unexpected cosmological contexts.

In practice, the effect found here is probably not large enough to solve the moduli problem. If so, theories with such scalar fields need to be excluded, which may serve as a useful constraint for string-inspired cosmological models.

III. Elastic scattering rate of heavy quarks

Overall picture in the heavy ion context

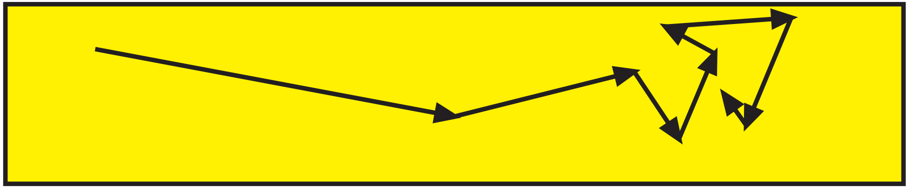
Heavy quarks are initially **produced** like in vacuum:



Subsequently they **propagate** through a thermal "medium".

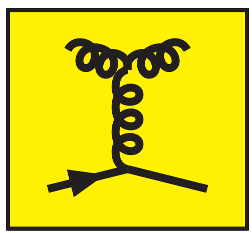
In the end they **decay**, often as $c \longrightarrow \ell \nu X$; the leptons ℓ can be observed experimentally.

Propagation through the medium



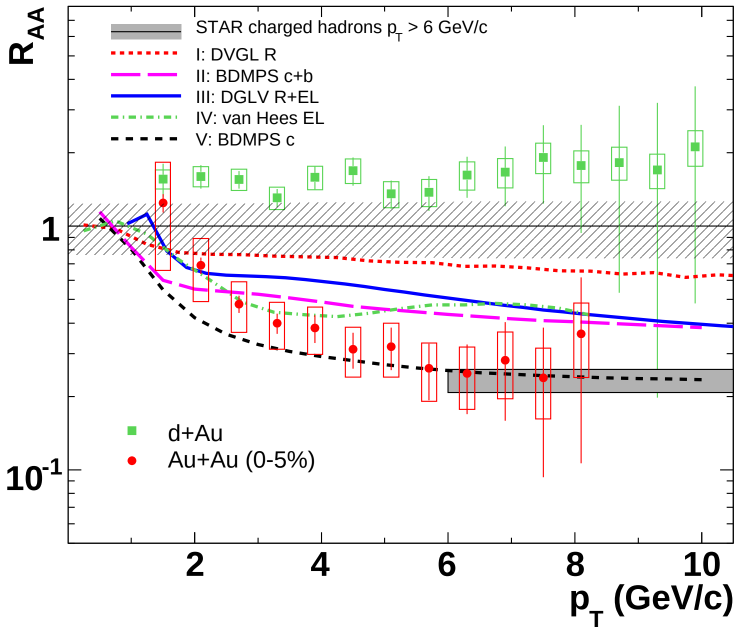
Like in Brownian motion, heavy quark jets (“open charm”) tend to get stopped (“quenched”) by scatterings.

Since the number of heavy quarks is conserved before decay, scatterings are of the **elastic** type from their point of view.



Indeed in Au + Au less ℓ observed than expected:

STAR nucl-ex/0607012, PHENIX nucl-ex/0611018



(Also observed is a non-zero v_2 associated with heavy quarks.)

Classical picture for a heavy quark already close to rest

Let p_i be the momentum. According to the Langevin equation,

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = -\eta\, p_i(t) + \xi_i(t) \ ,$$

$$\langle \xi_i(t)\xi_j(t')\rangle = \kappa\, \delta_{ij}\delta(t-t') \ , \qquad \langle \xi_i(t)\rangle = 0 \ .$$

Here $\kappa =$ "**momentum diffusion coefficient**",
and $\eta =$ "**drag coefficient**" $=$ "**kinetic thermalization rate**".

Fluctuation-dissipation relation

We can solve exactly for the time evolution:

$$p_i(t) = p_i(0)e^{-\eta t} + \int_0^t \mathrm{d}t' \, e^{\eta(t'-t)} \xi_i(t') \; .$$

In particular, letting the system thermalize by waiting,

$$\begin{aligned}
\langle p_i^2 \rangle_{\text{eq}} &\equiv \lim_{t\to\infty} \langle p_i^2(t) \rangle \\
&= \lim_{t\to\infty} \int_0^t \mathrm{d}t_1 \, e^{\eta(t_1-t)} \int_0^t \mathrm{d}t_2 \, e^{\eta(t_2-t)} \langle \xi_i(t_1) \xi_i(t_2) \rangle \\
&= \frac{\kappa}{2\eta} \; .
\end{aligned}$$

Equipartition tells that $\dfrac{\langle p_i^2 \rangle_{\text{eq}}}{2M_{\text{kin}}} = \dfrac{T}{2} \quad \Rightarrow \quad \eta = \dfrac{\kappa}{2TM_{\text{kin}}}.$

On the other hand, from the Langevin-equation,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\eta\, p_i + \xi_i \ , \quad \langle \xi_i(t)\xi_i(t')\rangle = \kappa\, \delta(t - t') \ ,$$

we note that κ can be obtained as

$$\kappa = \int_{-\infty}^{\infty} \mathrm{d}t\, \langle \xi_i(t)\xi_i(0)\rangle \ .$$

Moreover, we may identify ξ_i as the **Lorentz force**: for small velocities, $\xi_i = gE_i$, where E_i is the colour-electric field.

So,

$$\kappa_{\mathsf{cl}} = g^2 \int_{-\infty}^{\infty} \mathrm{d}t\, \langle E_i(t)E_i(0)\rangle \ .$$

Casalderrey-Solana Teaney hep-ph/0605199

In Quantum Mechanics, we replace E_i by a Heisenberg operator, $\hat{E}_i(t) \equiv e^{i\hat{H}t}\hat{E}_i(0)e^{-i\hat{H}t}$, and write down an ordering corresponding to the classical limit (symmetric in $t \to -t$):

$$\Delta_{\rm E}(t) \equiv \frac{g^2}{3N_{\rm c}} \sum_{i=1}^{3} {\rm Tr}_{N_{\rm c}} \left\langle \frac{1}{2} \left\{ \hat{E}_i(t), \hat{E}_i(0) \right\} \right\rangle_{\rm eq} ,$$

$$\langle \ldots \rangle_{\rm eq} = \frac{1}{\mathcal{Z}} {\rm Tr}[e^{-\beta\hat{H}}(\ldots)] , \quad \beta \equiv \frac{1}{T} .$$

Its Fourier transform is

$$\tilde{\Delta}_{\rm E}(\omega) \equiv \int_{-\infty}^{\infty} {\rm d}t \, e^{i\omega t} \, \Delta_{\rm E}(t) ,$$

and the momentum diffusion coefficient is $\kappa = \lim_{\omega \to 0} \tilde{\Delta}_{\rm E}(\omega)$.

In Quantum Field Theory, an important function describing the real-time dynamics is the **spectral function**, $\rho_{\mathrm{E}}(\omega)$, defined as

$$G_{\mathrm{E}}(t) \equiv \frac{g^2}{3N_{\mathrm{c}}} \sum_{i=1}^{3} \mathrm{Tr}_{N_{\mathrm{c}}} \left\langle \frac{1}{2} \left[\hat{E}_i(t), \hat{E}_i(0) \right] \right\rangle_{\mathrm{eq}} ,$$

$$\rho_{\mathrm{E}}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, C_{\mathrm{E}}(t) \; .$$

It can be shown that $\tilde{\Delta}_{\mathrm{E}}(\omega) = [1 + 2n_{\mathrm{B}}(\omega)] \, \rho_{\mathrm{E}}(\omega)$, where

$$n_{\mathrm{B}}(\omega) \equiv \frac{1}{\exp(\beta\omega) - 1} \overset{\omega \ll T}{\approx} \frac{T}{\omega} \; .$$

So:

$$\kappa = \lim_{\omega \to 0} \frac{2T \rho_{\mathrm{E}}(\omega)}{\omega} \; .$$

Proof:

More generally, *all* of the correlation functions defined above can be related to each other. In particular, all correlators can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. In order to do this, we may first insert sets of energy eigenstates into the definitions of $\Pi^{>}_{\alpha\beta}$ and $\Pi^{<}_{\alpha\beta}$:

$$\begin{aligned}
\Pi^{>}_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}}\int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}\mathrm{Tr}\Big[e^{-\beta\hat{H}+i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|}\hat{\phi}_\alpha(0,\mathbf{x})e^{-i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|}\hat{\phi}^\dagger_\beta(0,\mathbf{0})\Big] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}e^{(-\beta+it)E_m}e^{-itE_n}\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\,\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= \frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}}\sum_{m,n}e^{-\beta E_m}\,2\pi\,\delta(q^0+E_m-E_n)\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\,\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle\,, \qquad (0.1)\\
\Pi^{<}_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}}\int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}\mathrm{Tr}\Big[e^{-\beta\hat{H}}\underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|}\hat{\phi}^\dagger_\beta(0,\mathbf{0})e^{i\hat{H}t}\underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|}\hat{\phi}_\alpha(0,\mathbf{x})e^{-i\hat{H}t}\Big] \\
&= \frac{1}{\mathcal{Z}}\sum_{m,n}\int \mathrm{d}t\,\mathrm{d}^3\mathbf{x}\,e^{iQ\cdot x}e^{(-\beta-it)E_n}e^{itE_m}\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle\,\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \qquad (0.2)\\
&= \frac{1}{\mathcal{Z}}\int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}}\sum_{m,n}e^{-\beta E_n}\,2\pi\,\underbrace{\delta(q^0+E_m-E_n)}_{E_n=E_m+q^0}\langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle\,\langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= e^{-\beta q^0}\,\Pi^{>}_{\alpha\beta}(Q)\,. \qquad (0.3)
\end{aligned}$$

This is the Fourier-space version of the KMS relation. Consequently

$$\rho_{\alpha\beta}(Q) = \frac{1}{2}\,[\Pi^{>}_{\alpha\beta}(Q) - \Pi^{<}_{\alpha\beta}(Q)] = \frac{1}{2}\,(e^{\beta q^0}-1)\Pi^{<}_{\alpha\beta}(Q) \qquad (0.4)$$

and, conversely,

$$\begin{aligned}
\Pi^{<}_{\alpha\beta}(Q) &= 2n_\mathrm{B}(q^0)\rho_{\alpha\beta}(Q)\,, \qquad (0.5)\\
\Pi^{>}_{\alpha\beta}(Q) &= 2\frac{e^{\beta q^0}}{e^{\beta q^0}-1}\rho_{\alpha\beta}(Q) = 2[1+n_\mathrm{B}(q^0)]\rho_{\alpha\beta}(Q)\,, \qquad (0.6)
\end{aligned}$$

where $n_\mathrm{B}(x) \equiv 1/[\exp(\beta x)-1]$. Moreover,

$$\Delta_{\alpha\beta}(Q) = \frac{1}{2}\,[\Pi^{>}_{\alpha\beta}(Q) + \Pi^{<}_{\alpha\beta}(Q)] = [1+2n_\mathrm{B}(q^0)]\rho_{\alpha\beta}(Q)\,. \qquad (0.7)$$

Note that $1+2n_\mathrm{B}(-q^0) = -[1+2n_\mathrm{B}(q^0)]$, so that if ρ is odd in $Q\to -Q$, then Δ is even.

The formulae shown can be derived more systematically by employing an effective field theory relevant for heavy quarks, namely **HQET** (Heavy Quark Effective Theory)

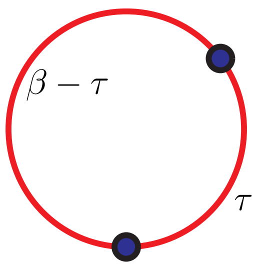
$$\Rightarrow \quad \eta = \frac{\kappa}{2 M_{\mathsf{kin}} T} \left(1 + O\left(\frac{\alpha_s^{3/2} T}{M_{\mathsf{kin}}} \right) \right) ,$$

$$\kappa = \lim_{\omega \to 0} \frac{2T \rho_{\mathrm{E}}(\omega)}{\omega} .$$

Here $\rho_{\rm E}$ is the spectral function corresponding to the Euclidean correlator

$$G_{\rm E}(\tau) = -\frac{1}{3}\sum_{i=1}^{3} \frac{\langle \mathrm{Re}\,\mathrm{Tr}[U_{\beta;\tau}\, gE_i(\tau,\mathbf{0})\, U_{\tau;0}\, gE_i(0,\mathbf{0})]\rangle}{\langle \mathrm{Re}\,\mathrm{Tr}[U_{\beta;0}]\rangle}\,,$$

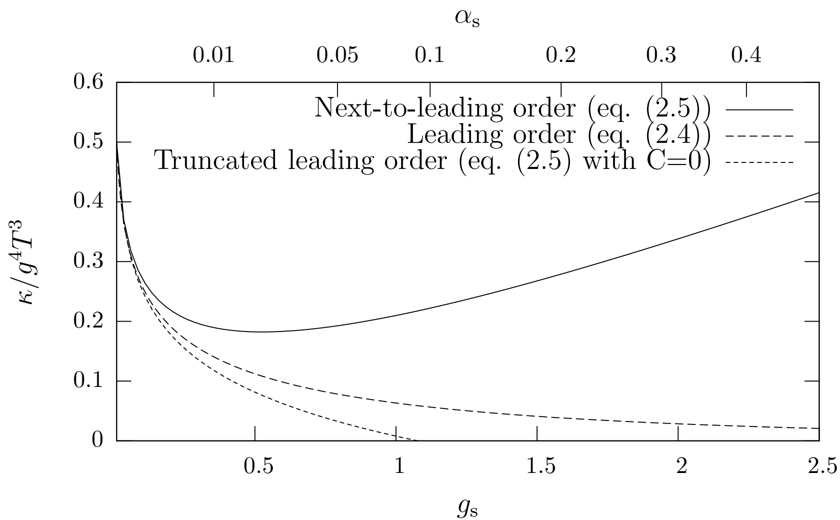
where $U_{\tau_b;\tau_a}$ is a Wilson line in the time direction.



Caron-Huot ML Moore 0901.1195

Phenomenological interlude

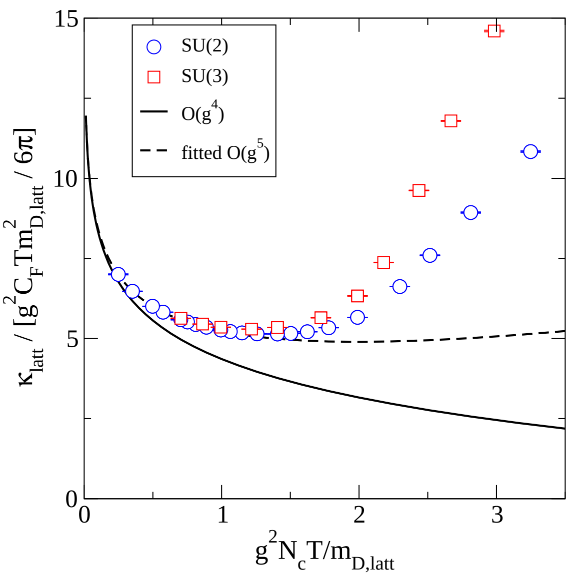
Within leading-order ($\mathcal{O}(g^4)$) and next-to-leading order ($\mathcal{O}(g^5)$; first ever NLO transport coefficient!) perturbation theory:



Caron-Huot Moore,
0708.4232; 0801.2173

There appears to be a huge correction in this case!

Within a non-perturbative framework called "classical lattice gauge theory", which is kind of an effective theory for real-time quantities in bosonic quantum field theory ($m_{\mathrm{D,latt}} \sim g^2 T/a$):



ML Moore Philipsen Tassler, 0902.2856

The real answer could be larger still!

Numerically, converting κ to $\eta = \dfrac{\kappa}{2TM_{\mathrm{kin}}}$:

$$\eta_{\mathrm{pQCD}} \sim \frac{g^4}{8\pi}\frac{T^2}{M_{\mathsf{kin}}} \sim 0.3\frac{T^2}{M_{\mathsf{kin}}}$$

Braaten Thoma PRD 44 (1991) 2625;
Moore Teaney hep-ph/0412346

$$\eta_{\mathrm{exp}} \sim (1\ldots3) \times \frac{T^2}{M_{\mathsf{kin}}}$$

e.g. Akamatsu et al 0809.1499

$$\eta_{\mathrm{AdS}} \sim \frac{g\pi\sqrt{3}}{2}\frac{T^2}{M_{\mathsf{kin}}} \sim 4\frac{T^2}{M_{\mathsf{kin}}}$$

Herzog et al hep-th/0605158;
Gubser hep-th/0605182;
Casalderrey Teaney hep-ph/0605199

So, we would "like" to have a large κ, but to be sure that the picture is correct we need to determine it from lattice-QCD!

Basics of spectral functions & Euclidean correlators

Heisenberg picture:

$$\hat{O}(t) = e^{i\hat{H}t}\hat{O}(0)e^{-i\hat{H}t} \ .$$

Spectral function:

$$\rho(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \frac{1}{\mathcal{Z}} \, \mathrm{Tr} \left\{ e^{-\beta\hat{H}} \frac{1}{2} \left[\hat{O}(t), \hat{O}(0) \right] \right\} \ .$$

Transport coefficient: $\lim\limits_{\omega\to 0} \dfrac{\rho(\omega)}{\omega}$.

Some gymnastics with Green's functions

Apart from the "real-time" Heisenberg-operators

$$\hat{A}(t) = e^{i\hat{H}t}\hat{A}(0)e^{-i\hat{H}t} ,$$

we define "imaginary-time" Heisenberg-operators as

$$\hat{A}(\tau) \equiv e^{\hat{H}\tau}\hat{A}(0)e^{-\hat{H}\tau} .$$

Despite the name, $\tau \in \mathbb{R}$; in fact, we always restrict to

$$0 \leq \tau \leq \beta .$$

Then we define:

$$\Pi_>(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \langle \hat{A}(t)\hat{A}(0)\rangle \ ,$$

$$\Pi_<(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \langle \hat{A}(0)\hat{A}(t)\rangle \ ,$$

$$\rho(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \left\langle \frac{1}{2}[\hat{A}(t), \hat{A}(0)] \right\rangle \ ,$$

$$G_E(\tau) \equiv \langle \hat{A}(\tau)\hat{A}(0)\rangle \ ,$$

$$\tilde{G}_E(\omega_n^{\mathsf{b}}) \equiv \int_0^{\beta} \mathrm{d}\tau \, e^{i\omega_n^{\mathsf{b}}\tau} G_E(\tau) \ ; \ \ \omega_n^{\mathsf{b}} \equiv 2\pi n T \ , \ \ n \in \mathbb{Z} \ .$$

Here the expectation value is $\langle \ldots \rangle \equiv \frac{1}{\mathcal{Z}} \operatorname{Tr} \left\{ e^{-\beta\hat{H}}(\ldots) \right\}$, and it is easy to see that $G_E(\beta) = G_E(0)$, i.e. $G_E(\tau)$ is periodic.

Apart from relations between the Minkowskian objects that we saw before, we need the following relations to the Euclidean ones:

$$\tilde{G}_E(\omega_n^{\mathsf{b}}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{\omega - i\omega_n^{\mathsf{b}}} \quad \text{“spectral representation”} \ ,$$

$$\rho(\omega) = \frac{1}{2i} \Big\{ \tilde{G}_E\Big(-i[\omega + i0^+]\Big) - \tilde{G}_E\Big(-i[\omega - i0^+]\Big) \Big\} \ ,$$

$$G_E(\tau) = \int_0^{\infty} \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \frac{\cosh\left(\frac{\beta}{2} - \tau\right)\omega}{\sinh\frac{\beta\omega}{2}} \ .$$

(The second one justifies the missing step on p.41, $\rho = \mathrm{Im}\, \tilde{G}_E$.)

Proof of the "spectral representation":

$$
\begin{aligned}
\tilde{G}_E(\omega_n^{\mathrm{b}}) &= \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n^{\mathrm{b}}\tau} \Big[\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, e^{-i\omega t}\Pi_>(\omega)\Big]_{it\to\tau} \\
&= \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n^{\mathrm{b}}\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-\omega\tau}\Pi_>(\omega) \\
&= \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n^{\mathrm{b}}\tau} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} e^{-\omega\tau} \frac{2e^{\beta\omega}}{e^{\beta\omega}-1}\rho(\omega) \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{1-e^{-\beta\omega}} \left[\frac{e^{(i\omega_n^{\mathrm{b}}-\omega)\tau}}{i\omega_n^{\mathrm{b}}-\omega}\right]_0^\beta \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{1-e^{-\beta\omega}} \frac{e^{-\beta\omega}-1}{i\omega_n^{\mathrm{b}}-\omega} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{\omega - i\omega_n^{\mathrm{b}}}\;.
\end{aligned}
$$

To prove the relation

$$\rho(\omega) \;=\; \frac{1}{2i}\left\{\tilde{G}_E\left(-i[\omega + i0^+]\right) - \tilde{G}_E\left(-i[\omega - i0^+]\right)\right\} \;,$$

start from the spectral representation

$$\tilde{G}_E(\omega_n^{\mathsf{b}}) \;=\; \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\pi} \frac{\rho(\omega)}{\omega - i\omega_n^{\mathsf{b}}} \;,$$

and make use of

$$\frac{1}{x \pm i0^+} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x) \;.$$

To prove the third relation, use the spectral representation:

$$
\begin{aligned}
G_E(\tau) &= T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \, \tilde{G}_E(\omega_n^{\mathsf{b}}) \\
&= T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \left[\frac{\rho(\omega)}{\omega - i\omega_n^{\mathsf{b}}} + \frac{\rho(-\omega)}{-\omega - i\omega_n^{\mathsf{b}}} \right] \\
&= T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \left[\frac{1}{\omega - i\omega_n^{\mathsf{b}}} + \frac{1}{\omega + i\omega_n^{\mathsf{b}}} \right] \\
&= \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \, T \sum_{\omega_n^{\mathsf{b}}} e^{-i\omega_n^{\mathsf{b}}\tau} \frac{2\omega}{\omega^2 + (\omega_n^{\mathsf{b}})^2} \\
&= \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho(\omega) \frac{\cosh\left(\frac{\beta}{2} - \tau\right)\omega}{\sinh \frac{\beta\omega}{2}} \, .
\end{aligned}
$$

We used $\rho(-\omega) = -\rho(\omega)$, true under weak assumptions.

To summarize, the spectral function $\rho(\omega)$ determines the Euclidean correlators, both in τ-space ($G_E(\tau)$) or in ω_n^{b}-space ($\tilde{G}_E(\omega_n^{\text{b}})$). The relations are **in principle** invertible.

This is conceptually comforting, because Euclidean observables can be computed with regular functional integrals, a well-defined procedure even on the non-perturbative (lattice) level.

Consider now a correlator of two conserved charges:

$$G^E_{00}(\tau) \equiv \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_0(\tau,\mathbf{x})\hat{J}_0(0,\mathbf{0}) \right\rangle \ .$$

The claim is that

$$\frac{\rho_{00}(\omega)}{\omega} \overset{!}{=} \Delta_{00}\beta\pi\delta(\omega) \ .$$

This is because the correlator must be independent of τ:

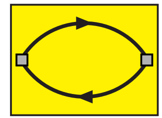
$$\begin{aligned} G^E_{00}(\tau) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\rho_{00}(\omega)\frac{\cosh\left(\frac{\beta}{2}-\tau\right)\omega}{\sinh\frac{\beta\omega}{2}} \\ &= \Delta_{00} \ . \end{aligned}$$

Indeed, the correlator of a conserved charge is necessarily a constant (thanks to a Ward-Takahashi identity):

$$\partial_\tau \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_0(\tau, \mathbf{x}) \hat{J}_0(0, \mathbf{0}) \right\rangle = 0 \ .$$

Consider then the correlator of the spatial components:

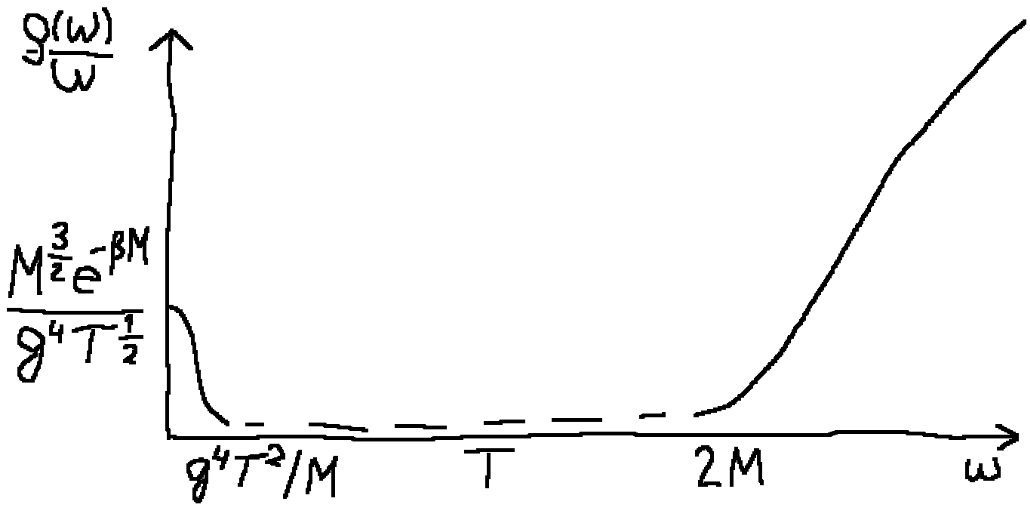
$$G^E_{ii}(\tau) \quad \equiv \quad \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_i(\tau,\mathbf{x})\hat{J}_i(0,\mathbf{0}) \right\rangle \; .$$

It turns out that in the free theory,  , this again contains a τ-independent "zero-mode":

$$\sum_{i=1}^{3} G^E_{ii}(\tau) = 4N_\mathrm{c}\frac{3T}{M}\left(\frac{MT}{2\pi}\right)^{3/2} e^{-\beta M} + (\tau - \text{dep.}) \; .$$

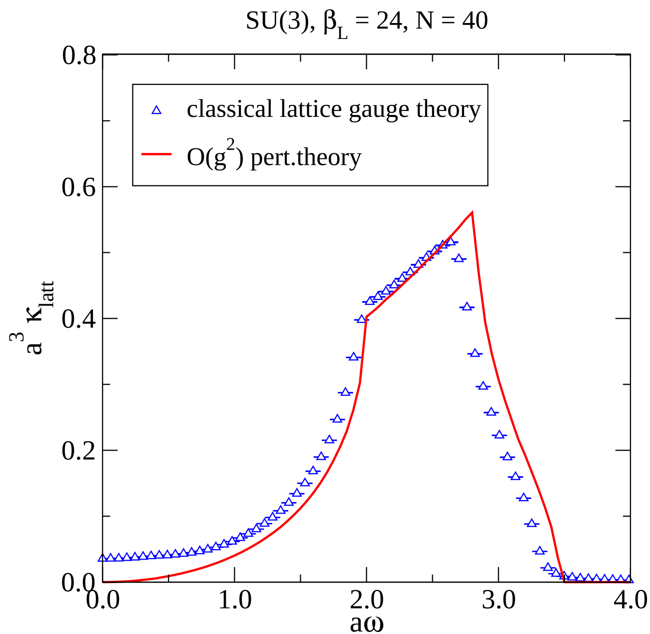
However, now interactions can smoothen $\delta(\omega)$ from $\rho_{ii}(\omega)/\omega$. This yields a "**transport peak**".

So, generically, spectral functions corresponding to spatial components of conserved currents have **transport peaks** at small frequencies. For instance, for $\hat{J}_i$ of heavy quarks:



This makes a direct lattice study of $\lim_{\omega\to 0}\frac{\rho(\omega)}{\omega}$ **very** difficult!

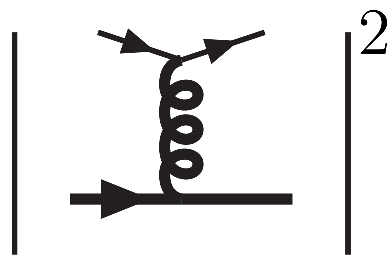
Fortunately, there is **no conservation law** associated with the **electric field correlator**, so the behaviour should be smoother. According to classical lattice gauge theory again:



Hopefully lattice data for $\kappa_{\rm QCD}$ will be available soon!

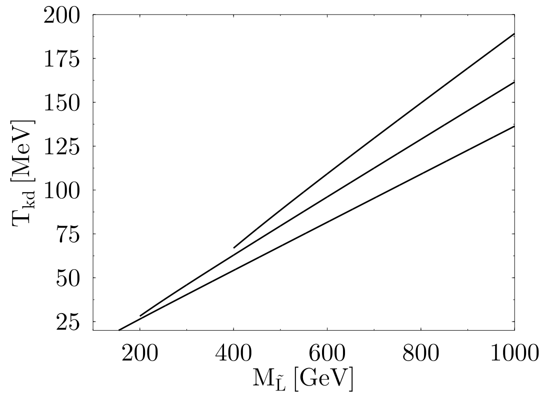
What does this have to do with cosmology?

Cold Dark Matter is by definition non-relativistic, and **kinetically** decouples when elastic scatterings, albeit with **weak interactions**, cease to be active.



Then their phase space density no longer behaves as $e^{-p^2/2M_{\mathsf{kin}}T}$. I.e. the **velocity dispersion** of dark matter, which may be visible in the large scale structures that can form, is determined by the temperature at which elastic scatterings last take place.

Typically, though, kinetic decoupling happens at a very low temperature, so probably a Fermi model treatment is sufficient, and then the analogy with a gauge theory like QCD is feeble.



Hofmann Schwarz Stöcker astro-ph/0104173

But at least a conceptual link exists, and could perhaps play a more significant role e.g. in connection with leptogenesis.

Conclusions

Classical arguments / effective theory techniques allow to reduce the heavy quark “momentum diffusion coefficient” to a gluonic observable, the electric field correlator (along a Polyakov loop).

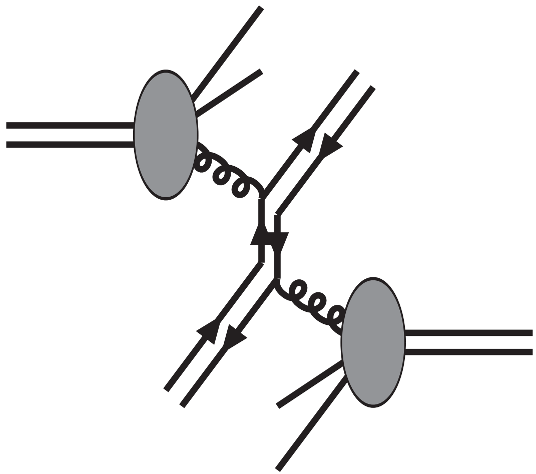
The convergence of the weak-coupling expansion depends on the observable; it appears to be very slow for κ.

Fortunately the electric field correlator might be more manageable on the lattice than those of conserved currents, so a non-perturbative determination of κ may be possible.

IV. Inelastic reactions: quarkonium in medium

For quarkonium we can imagine a chain of events similar to that for single quarks:

Initial **production** like in vacuum.

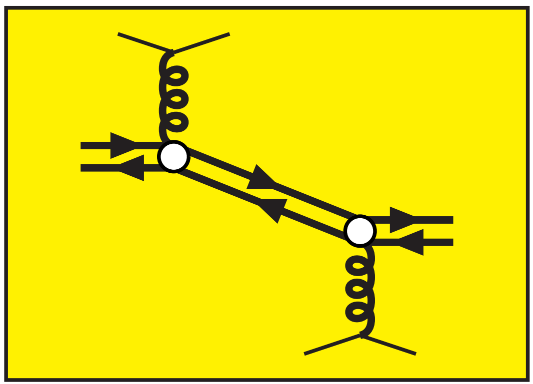


Subsequent **propagation** through a thermal "medium".

In the end a **decay**, often as $q\bar{q} \longrightarrow \ell^+\ell^-$.

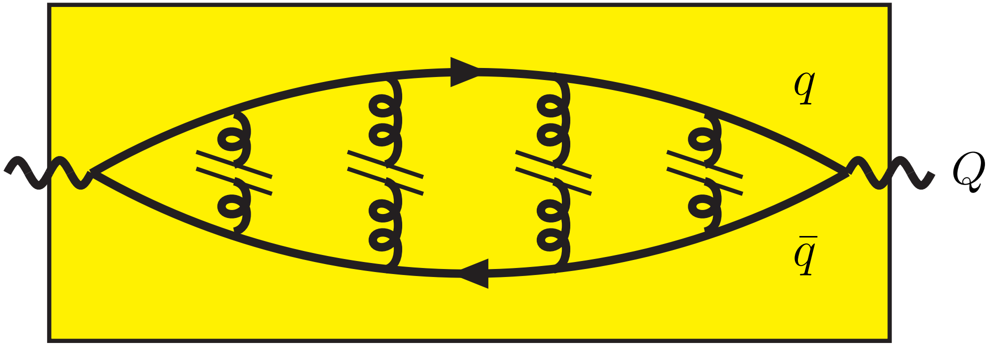
During **propagation** heavy quarkonium would feel less drag from elastic scatterings than heavy quarks because it has no net colour charge.

However, because of its finite size, it does have a **colour dipole** which leads to kicks and an eventual "decoherence" of its state:



But these effects should be suppressed by $\mathcal{O}(rT)^2$?

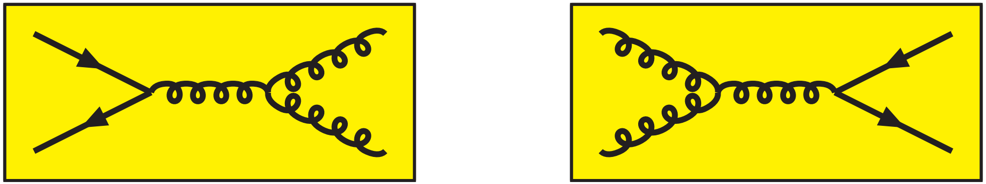
On the other hand the Coulomb potential gets Debye-screened within the medium:



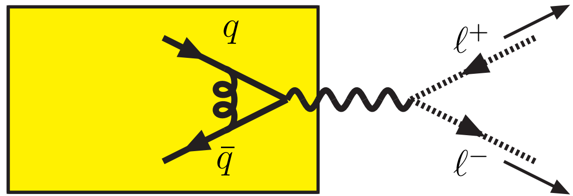
So the effective r could be larger than at $T = 0$, and the thermal effects on quarkonium propagation significant.

Once again these effects make themselves visible in the shape of the quarkonium **spectral function**.

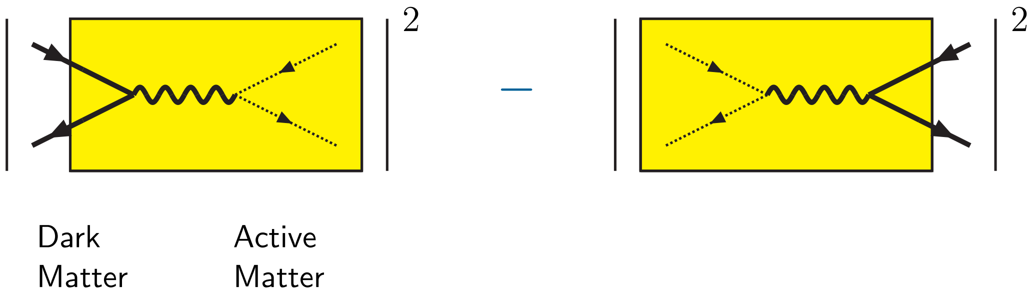
However, we could also imagine **inelastic** processes for quarkonium; both within a medium, amounting to **chemical thermalization**:



. . . as well as with decay products escaping the medium:



These processes are interesting because of some analogy with the inelastic dark matter processes that we saw before:



In the case $q\bar{q} \to \ell^+\ell^-$ the inelastic part is electromagnetic and thus purely perturbative! This allows to reduce the inelastic rate to the "elastic" spectral function (again a "Kubo relation"):

McLerran Toimela 1985; Weldon 1990; Gale Kapusta 1991

$$\frac{\mathrm{d}N_{\ell^+\ell^-}}{\mathrm{d}^4x\mathrm{d}^4Q} = \frac{-2e^4Z^2}{3(2\pi)^5Q^2}\Big(1+\frac{2m_\ell^2}{Q^2}\Big)\Big(1-\frac{4m_\ell^2}{Q^2}\Big)^{\frac{1}{2}} n_\mathrm{B}(q^0)\rho_\mathrm{V}(Q) \;;$$

$$\rho_\mathrm{V}(Q) \equiv \int_{-\infty}^{\infty}\mathrm{d}t \int \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \left\langle \frac{1}{2}[\hat{\mathcal{J}}^\mu(x), \hat{\mathcal{J}}_\mu(0)] \right\rangle \;.$$

For $q^0 \gtrsim 2M$ we see the Boltzmann suppression associated with dilepton production from thermalized heavy quarkonium.

However such a thermalization requires that inelastic reactions within the medium are fast enough to take place.

To keep in mind once again:

Propagation takes place in Minkowskian time (t), with a Minkowskian frequency (ω), but within a thermal system ($\beta = 1/T$).

$$\left\langle \frac{1}{2}[\hat{\mathcal{J}}^\mu(t,\mathbf{x}), \hat{\mathcal{J}}_\mu(0,\mathbf{0})] \right\rangle \; ,$$

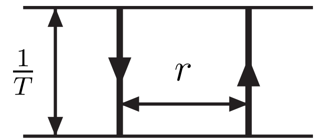
$$\hat{\mathcal{J}}^\mu(t,\mathbf{x}) = e^{i\hat{H}t}\hat{\mathcal{J}}^\mu(0,\mathbf{x})e^{-i\hat{H}t} \; ,$$

$$\langle \ldots \rangle \equiv \frac{1}{\mathcal{Z}}\,\mathrm{Tr}\left[(\ldots)e^{-\beta\hat{H}}\right] \; .$$

What is traditionally done in practice?

A popular lattice observable:

$$\psi_{\mathrm{C}}(r) \equiv \frac{1}{N_{\mathrm{c}}} \langle \mathrm{Tr}[P_{\mathbf{r}} P_{\mathbf{0}}^{\dagger}] \rangle_{\text{Coulomb}} \; .$$



No real time here! But empirically "nice".

(The same object in Landau gauge, or gauge-invariant alternatives, such as $\psi_{\mathrm{W}}(r) \equiv \frac{1}{N_{\mathrm{c}}} \langle \mathrm{Tr}[P_{\mathbf{r}} W_0 P_{\mathbf{0}}^{\dagger} W_0^{\dagger}] \rangle$ or $\psi_{\mathrm{T}}(r) \equiv \frac{1}{N_{\mathrm{c}}^2} \langle \mathrm{Tr}[P_{\mathbf{r}}]\, \mathrm{Tr}[P_{\mathbf{0}}^{\dagger}] \rangle$, do **not** reduce to the known zero-temperature static potential at short distances.)

Weak-coupling expression for $0 \lesssim rT \lesssim 1$ up to $\mathcal{O}(\alpha_s^2)$:

Burnier et al 0911.3480

$$\ln\left(\frac{\psi_{\rm C}(r)}{|\psi_{\rm P}|^2}\right) \approx \frac{g^2 C_F \exp(-m_{\rm E} r)}{4\pi T r}\left\{1 + \frac{g^2}{(4\pi)^2}\left[\frac{11 N_{\rm c}}{3}\Big(L_{\rm b}+1\Big) - \frac{2N_{\rm f}}{3}\Big(L_{\rm f}-1\Big)\right]\right\}$$

$$+ \frac{g^4 C_F N_{\rm c} \exp(-m_{\rm E} r)}{(4\pi)^2}\left[2 - \ln(2m_{\rm E} r) - \gamma_{\rm E} + e^{2m_{\rm E} r} E_1(2m_{\rm E} r)\right] - \frac{g^4 C_F N_{\rm c}}{(4\pi)^2}\frac{\exp(-2m_{\rm E} r)}{8T^2 r^2}$$

$$+ \frac{g^4 C_F N_{\rm c}}{(4\pi)^2}\left[\frac{1}{12T^2r^2} + \frac{{\rm Li}_2(e^{-4\pi T r})}{(2\pi T r)^2} + \frac{1}{\pi T r}\int_1^\infty {\rm d}x\left(\frac{1}{x^2} - \frac{1}{2x^4}\right)\ln\Big(1 - e^{-4\pi T r x}\Big)\right]$$

$$+ \frac{g^4 C_F N_{\rm f}}{(4\pi)^2}\left[\frac{1}{2\pi T r}\int_1^\infty {\rm d}x\left(\frac{1}{x^2} - \frac{1}{x^4}\right)\ln\frac{1 + e^{-2\pi T r x}}{1 - e^{-2\pi T r x}}\right] + \mathcal{O}(g^5)\ ,$$

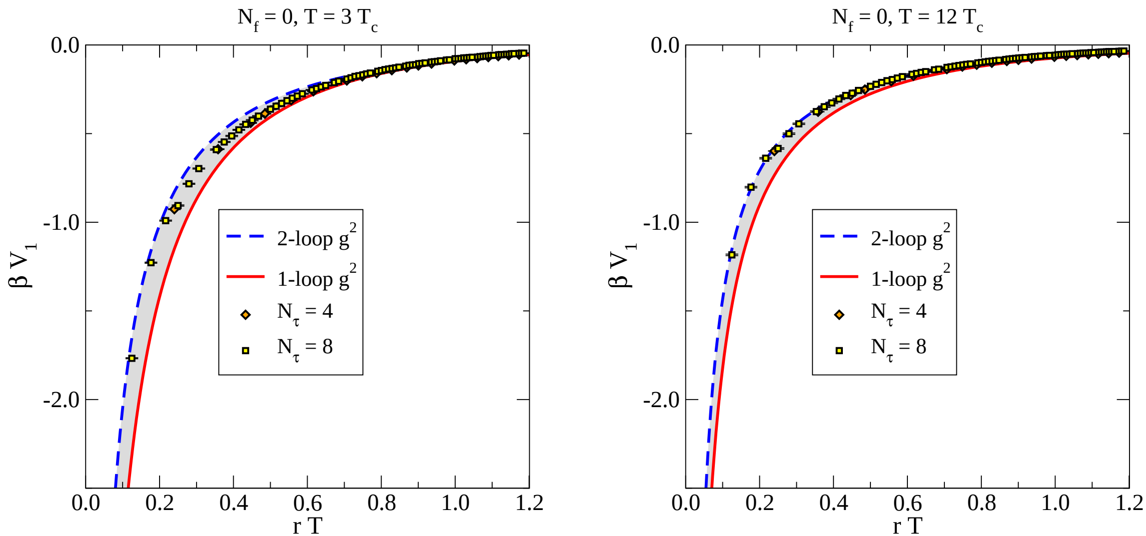
where $\psi_{\rm P}$ is the expectation value of a single Polyakov loop.

For $r \ll \frac{1}{T}$, this reproduces the classic $T = 0$ static potential.

Fischler NPB 129 (1977) 157

Comparison with lattice $[\beta V_1 \equiv -\ln(\psi_\mathrm{C}/|\psi_\mathrm{P}|^2)]$:

$N_\mathrm{f} = 0$ data from Kaczmarek et al hep-lat/0207002



So, reasonable agreement even at surprisingly low temperatures, **when computing the same gauge-fixed quantity**.

How to really address heavy quarkonium?

In order to get a handle on the many scales appearing, both vacuum and thermal, need once again to make use of **effective field theories**.

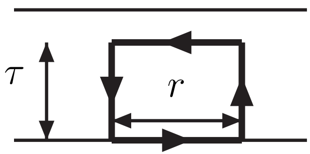
For quarkonium, the relevant framework is that of **NRQCD** (Non-Relativistic QCD) or one of its descendants (pNRQCD etc).

In such frameworks, various heavy quark potentials appear as **matching coefficients**, and can be given a concrete definition (at least within weak-coupling expansion, which now appears reasonable).

in the thermal context: ML et al hep-ph/0611300; Beraudo et al 0712.4394; Escobedo Soto 0804.0691; Brambilla et al 0804.0993

It is important to keep in mind that the Euclidean $\beta = 1/T$ is "small", while the Minkowskian t is "large" in the "static" limit.

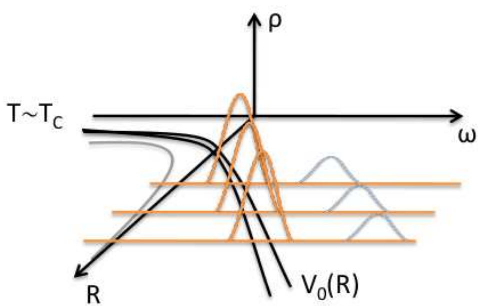
$\Rightarrow$ formally, define a potential from analytic continuation:



$$
\begin{aligned}
C_E(\tau, r) &\equiv \langle \mathrm{Tr}[W_E(\tau, r)] \rangle \ , \\
i\partial_t C_E(it, r) &\equiv V_>(t, r) C_E(it, r) \ .
\end{aligned}
$$

The static limit $V_>(\infty, r)$ through a spectral function $\rho(\omega, r)$?

Rothkopf Hatsuda Sasaki 0910.2321



Position of the spectral peak: average energy, $\mathrm{Re}\, V_>(\infty, r)$.
Its width: decoherence, $\mathrm{Im}\, V_>(\infty, r)$.

Explicitly at $\mathcal{O}(\alpha_s)$:

$$\mathrm{Re}\, V_{>}(\infty, r) = -\frac{g^2 C_F}{4\pi}\left[m_{\mathsf{E}} + \frac{\exp(-m_{\mathsf{E}} r)}{r}\right] ,$$

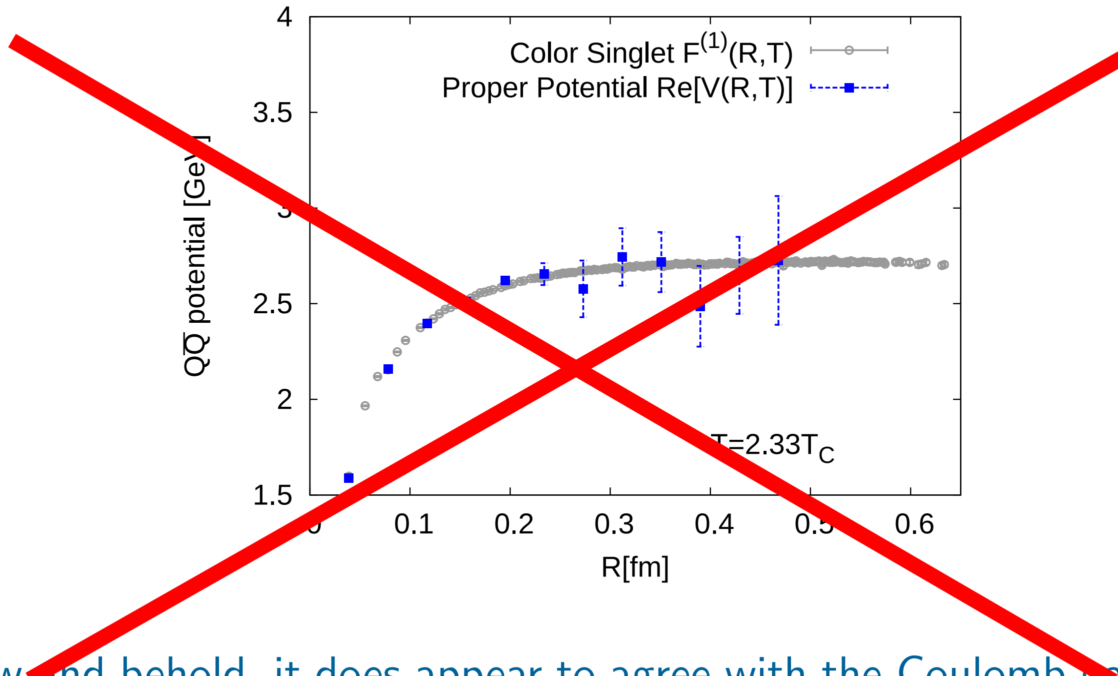
$$\mathrm{Im}\, V_{>}(\infty, r) = -\frac{g^2 T C_F}{4\pi}\,\phi(m_{\mathsf{E}} r) ,$$

where $m_{\mathsf{E}} \sim gT$ is the Debye mass, $C_F \equiv 4/3$, and

$$\phi(x) = 2\int_0^\infty \frac{\mathrm{d}z\, z}{(z^2+1)^2}\left[1 - \frac{\sin(zx)}{zx}\right] .$$

At $m_{\mathsf{E}} r \ll 1$, $\phi \sim (m_{\mathsf{E}} r)^2$, as is appropriate for a dipole.

Numerical test for the **real part**: Rothkopf Hatsuda Sasaki 0910.2321



Low and behold, it does appear to agree with the Coulomb gauge potential! But more precision required for definite conclusions.

Determine finally the spectral function and the dilepton rate

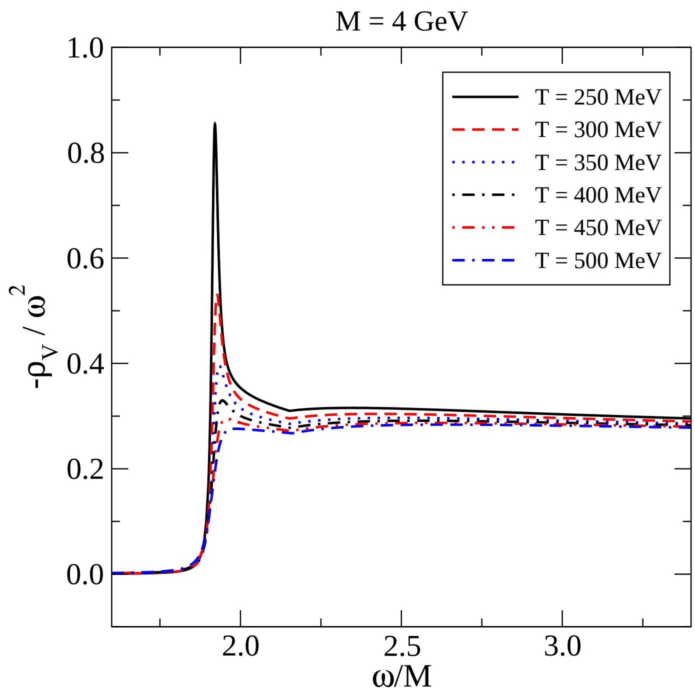
Insert $V_>(\infty, r)$ into the time-dependent Schrödinger:

$$\left\{ i\partial_t - \left[2M + V_>(\infty, r) - \frac{\nabla_{\mathbf{r}}^2}{M} + \mathcal{O}(\frac{1}{M^2}) \right] \right\} C_>(t) = 0 \; .$$

Solve and take Fourier transform from $C_>(t)$ to $\tilde{C}_>(\omega)$, with $\omega = q^0$, and finally convert to spectral function (cf. page 57)

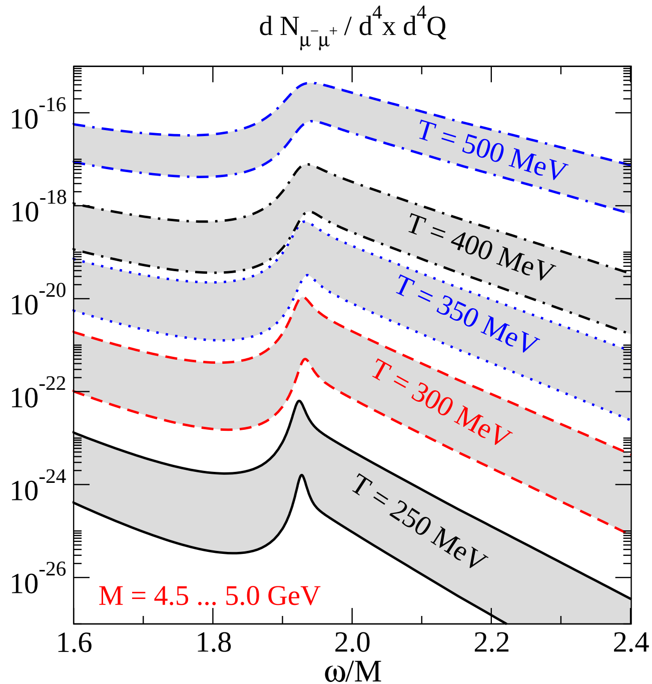
$$\rho_{\mathrm{V}}(Q) = \frac{1}{2}\left(1 - e^{-\beta q^0}\right) \tilde{C}_>(Q)$$

Result for the spectral function with such ingredients:



Burnier et al 0812.2105

Corresponding result for the thermal dilepton rate from $\bar{b}b$:



Burnier et al 0812.2105

As already mentioned, in cosmology a somewhat similar rate is relevant for Cold Dark Matter particles:

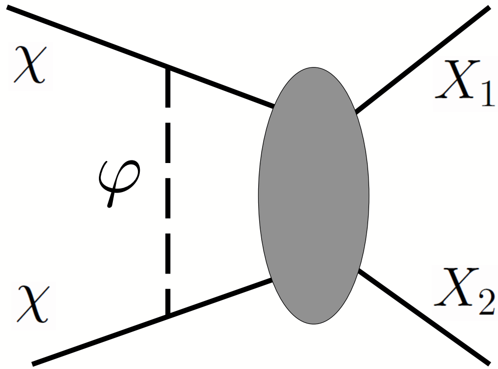


figure from Drees Kim Nagao 0911.3795

Another formal counterpart might be the initial chemical thermalization and/or final decoupling of heavy Majorana neutrinos relevant for leptogenesis.

Conclusions

Hot QCD is a simple but non-trivial theory for which systematic theoretical tools are being developed, and to some extent tested against heavy ion collisions as well as lattice QCD.

Some of these tools may find use also in cosmology, perhaps with the modification gluons $\longrightarrow$ grand unified gauge bosons, gluons $\longrightarrow W^{\pm}$, Z^0, or gluons $\longrightarrow$ Higgs.