QCD: finite baryon density; "sign problem"

Recall the global symmetry of (continuum) QCD:

$$SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1).$$

- $SU_L(N_f) \times SU_R(N_f)$: broken explicitly if $M \neq 0$, spontaneously if $M \to 0$.
- $U_A(1)$: broken by anomaly.
- $U_V(1)$ → **Unbroken exact global symmetry!**

The Noether theorem: existence of a global symmetry implies existence of a conserved current!

$U_V(1)$: $\quad \psi \to e^{i\alpha_V}\psi, \quad \bar\psi \to e^{-i\alpha_V}\bar\psi$.

For $\mathcal{L} = \bar\psi\left[\gamma_\mu D_\mu + M\right]\psi$ (i.e. continuum), the corresponding Noether current is

$$j^\mu \propto \frac{\partial \mathcal{L}_M}{\partial(\partial_\mu\psi)}\frac{\delta\psi}{\delta\alpha_V} = \frac{\partial}{\partial(\partial_\mu\psi)}\left[\bar\psi i\gamma^\mu\partial_\mu\psi\right]\cdot i\psi = -\bar\psi\gamma^\mu\psi$$

($\mathcal{L}_M$: Minkowskian)

In Euclidean space:

$$j^\mu \equiv \bar\psi\gamma_\mu\psi$$, where an overall sign has been chosen.

The zeroth component, "charge density":

$$j_0 = \bar u\gamma_0 u + \bar d\gamma_0 d + \bar s\gamma_0 s$$

$\bar u\gamma_0 u$ is interpreted as quark number density (quarks minus antiquarks)

$\frac{2}{3}e\,\bar u\gamma_0 u$ — " — electric charge density.

j_0 counts the total number of quarks. Since there are three quarks in a baryon, we define the <u>baryon number</u> B as

$$B = \frac{1}{3}\int d^3\bar x\, j_0 = \frac{1}{3}\int d^3\bar x\, \bar\psi\gamma_0\psi.$$

The baryon number is exactly conserved within QCD.
(Hadrons: mesons $\equiv \{B=0\}$ + baryons $\equiv \{B\neq 0\}$.)

Systems with a finite baryon number density are of some physical interest:

- inside heavy nuclei : nuclear matter, nuclear liquid
- inside neutron stars : baryon densities much higher still : protons & neutrons overlap, "quark matter" ?

It would be important to know the properties ["equation of state"; $p(T,\mu)$] of such matter, since they enter the Tolman-Oppenheimer-Volkov equation determining the structure of these objects.

A system with a finite baryon number density is often easier to describe by carrying out a Legendre transform and going into a system with a finite chemical potential :

$$\frac{\partial F(B,T,V)}{\partial B} \equiv \mu_B \quad ; \quad \Omega(\mu_B,T,V) \equiv F - \mu_B B$$

Baryon number (density) can the be recovered by

$$\frac{\partial \Omega(\mu_B,T,V)}{\partial \mu_B} = -B \quad ; \quad F(B,T,V) = \Omega + \mu_B B .$$

Stat. mech. :

$$\Omega(\mu_B,T,V) = -V p(\mu_B,T) = -T \ln \mathrm{Tr} \exp\left[-\beta(\hat{H} - \mu_B \hat{B})\right] .$$

($\hat{H}$: Hamiltonian; $\hat{B}$: baryon number)

Since B is conserved, $[\hat{H},\hat{B}]=0$, we can consider $\mu_B \hat{B}$ simply as a part of the Hamiltonian! Therefore also path integral can be derived as usual. Denoting $\mu = 3 \cdot \mu_B$, and recalling $\beta \to \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau$,

$$\mathcal{L} \to \bar{\Psi}\left[\gamma_\mu D_\mu - \mu\gamma_0 + M\right]\Psi$$

How can this system be discretised?

[P. Hasenfratz, F. Karsch, Phys. Lett. B 125 (1983) 308]

Recall (p. 71-72):

$$U_\mu = e^{iag_0 A_\mu} \quad , \quad A_\mu = A_\mu^a T^a$$

$$\text{(p. 93)} \Rightarrow S = \sum_{\bar{x}} a^4 \Big\{ \bar\Psi \left(M + \frac{4r}{a}\right)\Psi - \frac{1}{2a}\sum_\mu \Big[\bar\Psi(\bar{x}+a\hat\mu)\, U_\mu^\dagger(\bar{x})(r+\gamma_\mu)\,\Psi(\bar{x}) + \bar\Psi(\bar{x})\, U_\mu(\bar{x})(r-\gamma_\mu)\,\Psi(\bar{x}+a\hat\mu) \Big] \Big\}$$

In continuum limit:
* terms with r are $\mathcal{O}(a)$ (p. 91)
* the rest (apart from the mass term):

$$-\frac{1}{2a}\sum_\mu \Big[\bar\Psi(\bar{x}+a\hat\mu)\, U_\mu^\dagger(\bar{x})\gamma_\mu \Psi(\bar{x}) - \bar\Psi(\bar{x})\, U_\mu(\bar{x})\gamma_\mu \Psi(\bar{x}+a\hat\mu) \Big]$$

$$\approx -\frac{1}{2a}\sum_\mu \Big[\bar\Psi(\bar{x}+a\hat\mu)\gamma_\mu \Psi(\bar{x}) - \bar\Psi(\bar{x})\gamma_\mu \Psi(\bar{x}+a\hat\mu) + \bar\Psi(\bar{x})\,(-iag_0 A_\mu)\,\gamma_\mu \Psi(\bar{x}) - \bar\Psi(\bar{x})\,(iag_0 A_\mu)\,\gamma_\mu \Psi(\bar{x}) \Big]$$

Summing over $\bar{x}$ $\Rightarrow$

$$= \sum_\mu \bar\Psi(\bar{x})\gamma_\mu \frac{\Psi(\bar{x}+a\hat\mu) - \Psi(\bar{x}-a\hat\mu)}{2a} + \bar\Psi(\bar{x})\gamma_\mu i g_0 A_\mu \Psi(\bar{x})$$

$$\approx \bar\Psi(\bar{x})\gamma_\mu \left[\partial_\mu + i g_0 A_\mu \right] \Psi(\bar{x}) .$$

Now we note that the addition of a chemical potential corresponds to

$$i g_0 A_0 \rightarrow i g_0 A_0 - \mu \cdot \mathbb{1}_{N_c \times N_c}$$

Thus, on the lattice,

$$\boxed{\begin{aligned} U_0 &\rightarrow e^{-a\mu}\, U_0 \\ U_0^\dagger &\rightarrow e^{+a\mu}\, U_0 \end{aligned}}$$.

The "sign" problem

Let us look at the Dirac operator again.
For simplicity, we work in the continuum.

$$\not{D} + M - \mu\gamma_0 = \underbrace{\gamma_\mu\left[\partial_\mu + ig_0 A^a_\mu T^a\right]}_{\equiv D_0} + M \underbrace{- \mu\gamma_0}_{\equiv D_\mu} \quad ; \text{ indices suppressed; } \gamma_\mu^\dagger = \gamma_\mu .$$

Let us assume M to be degenerate for simplicity, $M = \mathrm{diag}\,(m, m, m)$.
Then it is enough to consider one flavour only.

What can we say about the eigenvalues?

(1) $(D_0 + D_\mu)\,v = \lambda v \quad \Rightarrow \quad (D_0 + m + D_\mu)\,v = (\lambda + m)\,v \quad \Rightarrow$ m plays a trivial role.

(2) $(D_0 + D_\mu)\,v = \lambda v \quad \Rightarrow \quad (D_0 + D_\mu)\,\gamma_5 v = -\gamma_5 (D_0 + D_\mu)\,v$

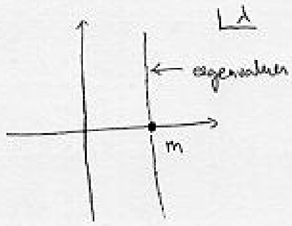
$= -\lambda \gamma_5 v \quad \Rightarrow$ eigenvalues come in pairs, λ & $-\lambda$, for $m=0$.

(3) $D_0^\dagger = -D_0 \quad \Rightarrow \quad \lambda^* = (v^\dagger D_0 v)^\dagger = v^\dagger D_0^\dagger v$

$= -v^\dagger D_0 v = -\lambda$

$\Rightarrow$ for $\mu = 0$, $m = 0$, they are purely imaginary!

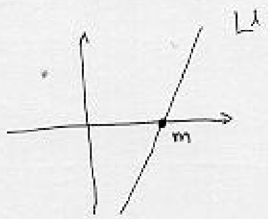
$\mu = 0$



$$\Rightarrow \quad \mathrm{Det}\,[D_0 + m] = (m + \lambda_1)(m + \lambda_1^*)(m + \lambda_2)(m + \lambda_2^*) \cdots \quad \in \mathbb{R}, \quad > 0 .$$

(4) $D_\mu^\dagger = D_\mu$

$\Rightarrow$ for $\mu \neq 0$, eigenvalues no longer purely imaginary for $m = 0$.



$\Rightarrow \mathrm{Det}\,[D_0 + m + D_\mu] \in \mathbb{C}$

$\Rightarrow$ weight $\mathrm{Det}\,[D_0 + m + D_\mu]\,\exp(-S^{(\text{gauge})}) \in \mathbb{C}$; can be positive or negative or oscillatory

Summary

- For applications in astrophysics, it would be interesting to compute the pressure of QCD as a function of the baryon chemical potential (even at $T = 0$).

- The baryon chemical potential can easily be included in the QCD action.

- The inclusion of the chemical potential makes, however, the weight of the functional integral in general complex. This turns out to be a major problem, since importance sampling ceases to work.

  $\Rightarrow$ some fresh ideas needed!

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