QCD: finite temperature physics; phase diagram

We have seen that like in $O(N)$ spin models, the ground state of QCD has a spontaneously broken global symmetry in the limit $M \to 0$: (p. 108, 109)

$$SU_L(N_f) \times SU_R(N_f) \to SU_V(N_f) .$$

Question: like the spontaneously broken symmetry in $O(N)$ spin models, does the one in QCD also get restored, if we go to high temperatures?

Answer: Yes! $T_c \sim 170$ MeV $\sim$ QCD-scale. [1eV = 11 600 K] [inside the sun: $T \sim 1$keV]

(Apart from "chiral symmetry restoration", the transition is also said to be associated with "deconfinement".)

[J.C. Collins & M.J. Perry, Phys. Rev. Lett. 34 (1975) 1353
N. Cabibbo & G. Parisi, Phys. Lett. B 59 (1975) 67.]

⇨ It would be important to determine the precise properties of this phase transition, because:

- temperatures as high as (and higher than) T_c appear in cosmology! Basic relation: $\frac{T}{\text{MeV}} \sim \frac{1}{\sqrt{t/s}}$, where t = age of the Universe. If the whole Universe went through a phase transition, that might have important implications.

- the same kind of temperatures can be reached momentarily in "heavy ion collision experiments". Brookhaven National Laboratory, Long Island, USA:

  $^{197}\text{Au} \longrightarrow \longleftarrow {}^{197}\text{Au}$

  Each particle has $E \sim 100 \frac{\text{GeV}}{\text{nucleon}}$.

Why does one expect a phase transition?

Consider minus the (grand canonical) free energy density, or pressure:

$$p(T,\mu) = -\frac{\Omega(V,T,\mu)}{V}, \quad \text{for } V \to \infty .$$

For massless relativistic degrees of freedom, and $\mu = 0$ (see next lecture!), the Stefan-Boltzmann law states that

$$p(T) \equiv p(T,0) = \frac{\pi^2}{90} T^4 \left\{ g_b + \frac{7}{8} g_f \right\},$$

where
$$\begin{cases} g_b = \text{number of bosonic degrees of freedom,} \\ g_f = \text{number of fermionic degrees of freedom.} \end{cases}$$

For small temperatures: $m_{\text{Goldstone}} \lesssim T \ll$ QCD-scale

$$\Rightarrow \begin{cases} g_b = N_f^2 - 1 = 8 \quad , \text{ if } N_f = 3 . \\ g_f = 0 \end{cases}$$

For large temperatures: $T \gg$ QCD - scale

$\Rightarrow$ "asymptotic freedom" (cp. 75)

$\Rightarrow$ quarks & gluons become weakly interacting

$$\Rightarrow \begin{cases} g_b = \underbrace{2}_{\text{polarisations of massless vector bosons}} \cdot \underbrace{(N_c^2 - 1)}_{\text{colours}} \\ g_f = \underbrace{4}_{\text{spinor indices, or spin-}\frac{1}{2}\text{ quark \& anti-quark}} \cdot \underbrace{N_c}_{\text{colours}} \cdot \underbrace{N_f}_{\text{flavours}} \end{cases}$$

$$\Rightarrow \text{tot} = 16 + \frac{7}{8} \cdot 4 \cdot 3 \cdot 3 = 47\frac{1}{2} \gg 8 .$$

Thus the behaviour of $p(T)$ changes significantly as the temperature T crosses the QCD-scale $\Rightarrow$ there could be a singularity in between.

How to determine the thermodynamics on the lattice?

p. 62 : $$Z = e^{-\frac{\Omega}{T}} = \int_{\phi(0,\bar{x}) = \phi(\beta\hbar,\bar{x})} \mathcal{D}\phi \, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3\bar{x} \, \mathcal{L}_{\text{Euclidean}}}$$

$$\Rightarrow \quad p(T) = \lim_{V\to\infty} \frac{T}{V} \ln Z$$

The path integral itself is, however, not directly measurable — $\underbrace{\text{importance sampling}}_{\text{p. 35 ff.}}$ works for expectation values of the type $\langle O \rangle$!

But the derivative can in principle be measured! In pure gauge theory, for instance, $Z = Z(\beta_G(a), N_\tau, N_x, N_y, N_z)$, with $\beta\hbar = \frac{1}{T} = N_\tau a$.

So keeping N_τ fixed,

$$\frac{d}{dT} Z = \frac{\partial Z}{\partial \beta_G} \cdot \frac{d\beta_G}{da} \cdot \left(-\frac{1}{N_\tau T^2} \right) = \frac{\partial Z}{\partial \beta_G} \cdot \frac{d\beta_G}{d\ln a} \left(-\frac{1}{T} \right) ,$$

$$\frac{\partial \ln Z}{\partial \beta_G} = \left\langle -\frac{1}{2N_c} \sum_x \sum_{\mu,\nu} \text{Tr}\left(\mathbb{1} - P_{\mu\nu} \right) \right\rangle = -\frac{1}{2N_c} \cdot N_\tau N_x N_y N_z \cdot \left\langle \sum_{\mu,\nu} \text{Tr}\left(\mathbb{1} - P_{\mu\nu} \right) \right\rangle$$

$$\Rightarrow \quad T \frac{dp}{dT} = p + \frac{T^2}{V} \cdot \left(-\frac{1}{T} \right) \cdot \left(-N_\tau N_x N_y N_z \right) \cdot \left\langle \frac{1}{2N_c} \sum_{\mu,\nu} \text{Tr}\left[\mathbb{1} - P_{\mu\nu} \right] \right\rangle \cdot \frac{d\beta_G}{d\ln a}$$

$$\Rightarrow \quad T^2 \frac{d}{dT}\left(\frac{p}{T} \right) = (N_\tau T)^4 \, \frac{d\beta_G}{d\ln a} \left\langle \frac{1}{2N_c} \sum_{\mu,\nu} \text{Tr}\left[\mathbb{1} - P_{\mu\nu} \right] \right\rangle . \qquad (*)$$

If $M \neq 0$, then at very low temperatures $T \ll m_\pi$ we can apply non-relativistic Boltzmann statistics :

$$p(T) \propto m_\pi^{\frac{3}{2}} T^{\frac{5}{2}} \exp\left(-\frac{m_\pi}{T} \right) \times \left(N_f^2 - 1 \right) .$$

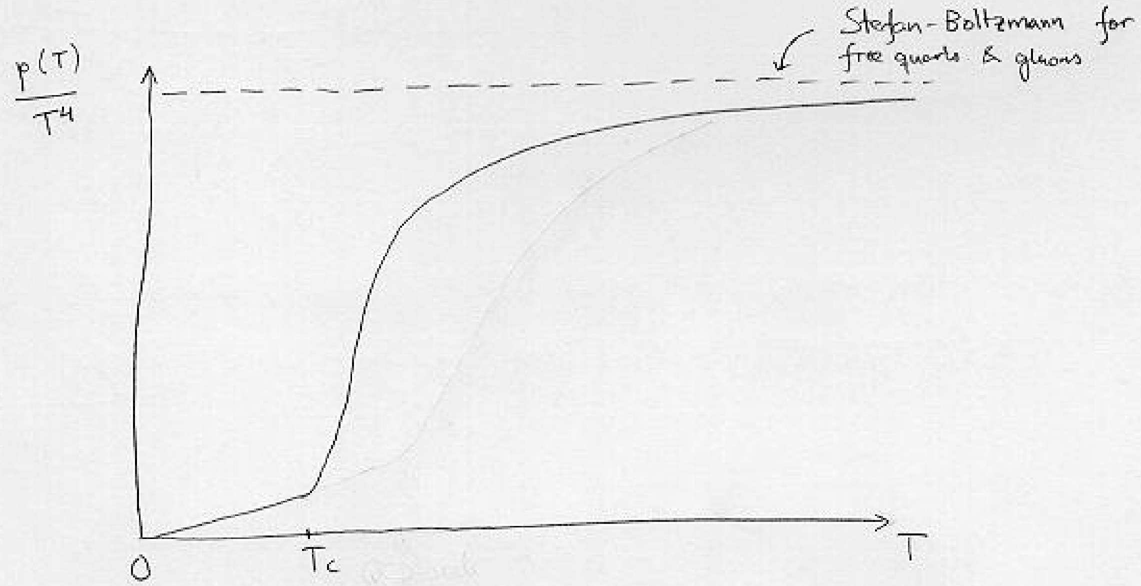
$\hookrightarrow$ Use this at some T_0 and integrate upwards.

If $M = 0$, use Stefan-Boltzmann for Goldstones at T_0.

Other relations analogous to $(*)$ can also be written down, for other thermodynamic functions.

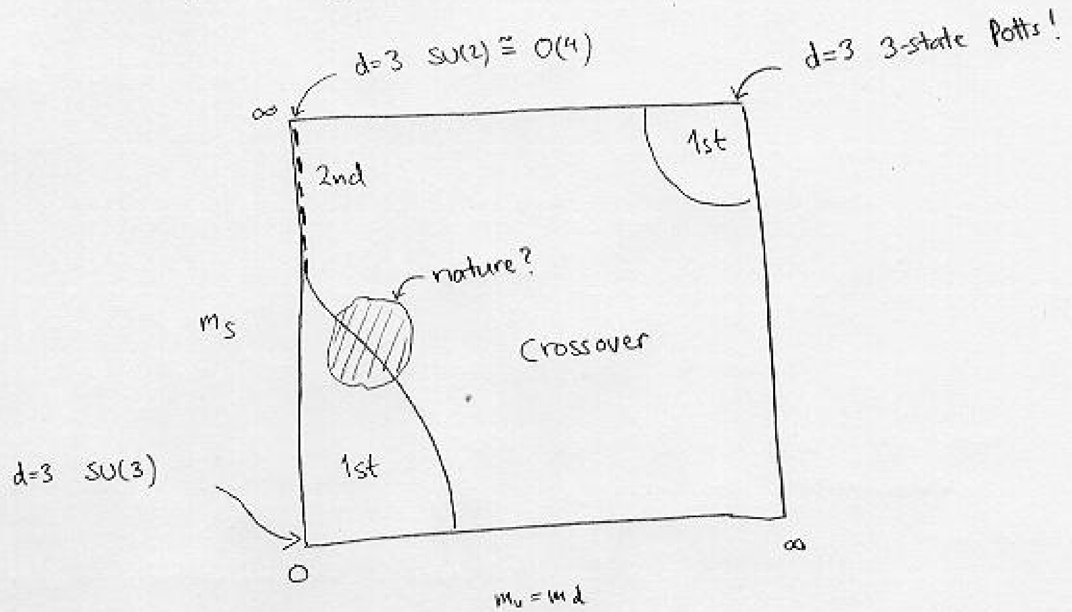
Results

Pressure:



What happens at "T_c"?

Is there an actual phase transition? (p. 8)
If yes, of which order?
Due to universality (p. 10), this depends strongly on global symmetries and thus, again, on the mass matrix M.



Summary:

- It is expected that the spontaneously broken chiral symmetry gets restored at a temperature of the order of the QCD scale.

- This behaviour can be monitored, for instance, by measuring (the derivative of) the pressure directly on the lattice.

- The pressure approaches a Stefan-Boltzmann law for free quarks and gluons, as $T \to \infty$.

- Whether there is a singularity $\equiv$ phase transition on the way, depends strongly on M. Again "effective theories", or universality in this case, can tell something. However quantitatively our $S^{(eff)}$ is no longer valid, since energy $\approx T \approx$ QCD-scale, so for precise results need full QCD simulations.

- More in some course on finite-temperature field theory...

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