QCD: spectroscopy, indirectly: matching to low-energy effective theory

We saw last time that for systems with a scale hierarchy (such as QCD in the light hadron sector), the straightforward lattice approach is doomed to fail. The same is true in many other analogous contexts as well.

On the other hand, in such cases it is often possible to make use of the scale hierarchy to construct an effective theory for the light degrees of freedom, like we did for the almost-Goldstone bosons in QCD.

A successful general philosophy:

* use the effective theory to predict the physical properties of the theory (such as light meson masses in QCD)

* use the original theory to determine the parameters of the effective theory ($F^2, \Sigma, \ldots$).

* the latter step ("matching") can often be carried out at volumes much <u>smaller</u> than needed for the former step, and should thus be better accessible.

$\Rightarrow$ let us see how it goes...

Meson masses in terms of F^2, Σ

$$S^{(eff)} = \int d^4x \left\{ \frac{F^2}{4} \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^+\right] - \frac{\Sigma}{2} \mathrm{Tr}\left[U M^+ + M U^+\right] + \dots \right\}.$$

For simplicity, let us take $m_u = m_d \equiv m$, so that $M = \begin{pmatrix} m & & \\ & m & \\ & & m_s \end{pmatrix}$.

$U \in SU(3)$.

Rather than $U = e^{i\omega \cdot T^a}$, we can write it directly in the "meson basis": $U \equiv \exp\left(\frac{2i\xi}{F}\right)$, $\xi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} \end{pmatrix}$;

$\pi^- = (\pi^+)^*$, $K^- = (K^+)^*$, $\bar{K}^0 = (K^0)^*$

To find out the masses, need to expand $S^{(eff)}$ to second order:

- $\frac{F^2}{4} \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^+\right] \Rightarrow \mathrm{Tr}\left[\partial_\mu \xi \partial_\mu \xi\right]$

$$= \frac{1}{2}\partial_\mu \pi_0 \partial_\mu \pi_0 + \frac{1}{2}\partial_\mu \eta \partial_\mu \eta + \partial_\mu \pi^+ \partial_\mu \pi^- + \partial_\mu K^+ \partial_\mu K^- + \partial_\mu K^0 \partial_\mu \bar{K}^0$$

- $-\frac{\Sigma}{2} \mathrm{Tr}\left[M(U+U^+)\right] \Rightarrow -\frac{\Sigma}{2} \cdot \left(-\frac{2}{F^2}\right) \cdot 2 \cdot \mathrm{Tr}\left[M \cdot \xi^2\right]$

$$= \frac{\Sigma}{F^2} \left\{ \frac{1}{2} \cdot 2m \cdot \pi_0^2 + \frac{1}{2} \cdot \frac{2}{3}(m+2m_s)\eta^2 + 2m\pi^+\pi^- + (m+m_s)K^+K^- + (m+m_s)K^0\bar{K}^0 \right\}$$

Recall that $\left\{ \begin{array}{l} S = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 \right\}, \phi \in \mathbb{R} \\ S = \int d^4x \left\{ \partial_\mu\phi^*\partial_\mu\phi + m^2\phi^*\phi \right\}, \phi \in \mathbb{C} \end{array} \right\} \Rightarrow \xi^{-1} = \text{mass} = m$

$$\Rightarrow \quad m_{\pi^0} = m_{\pi^\pm} = \sqrt{\frac{2\Sigma m}{F^2}} \qquad (\text{p. 101: } \sim 135\ \text{MeV})$$

$$m_{K^0,\bar{K}^0} = m_{K^\pm} = \sqrt{\frac{\Sigma(m+m_s)}{F^2}} \qquad \sim 495\ \text{MeV})$$

$$m_\eta = \sqrt{\frac{2\Sigma(m+2m_s)}{3F^2}} \qquad \sim 550\ \text{MeV})$$

First surprise: m_π, m_K are <u>not</u> $\sim m_u+m_d, m_u+m_s$, but with $\sqrt{\dots}$!!

Ratios:

$$\frac{m_K}{m_\pi} = \sqrt{\frac{1+\frac{m_s}{m}}{2}} = \frac{495}{135} \Rightarrow \frac{m_s}{m} \approx 26$$

$$\frac{m_\eta}{m_\pi} = \sqrt{\frac{1+\frac{2m_s}{m}}{3}} = \frac{550}{135} \Rightarrow \frac{m_s}{m} \approx 24$$

~ Ok!

To obtain absolute predictions, we need to determine F, Σ (or at least Σ / F^2).

- a "phenomenological" approach: by computing more observables with the effective theory (scatterings of particles, etc), F, Σ can also be estimated.

  Values: $F \approx 93$ MeV, $\Sigma \approx (250 \text{ MeV})^3$ $\Rightarrow$ $m \approx \frac{m_\pi^2 F^2}{2\Sigma} \approx \underline{5 \text{ MeV}}$!

  Much smaller than we expected!

- a hard test of QCD is to determine F, Σ with lattice!

Possibilities:

$*$ $$-\Sigma \equiv \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \left\langle \int d^4x \, \bar{u}(x) u(x) \right\rangle .$$

$*$ While masses are determined from the exponential falloff of two-point correlation functions, F^2 can be determined from the overall magnitude! It can be shown, for instance, that on a lattice with time extent L_τ,

$$\int d^3\bar{x} \left\langle [\bar{\psi}_L \gamma_0 T^a \psi_L](\bar{x}, t) \, [\bar{\psi}_L \gamma_0 T^b \psi_L](\bar{0}, 0) \right\rangle$$

$$\approx \mathrm{Tr}\,[T^a T^b] \, \frac{m_\pi F^2}{2} \cdot \underbrace{\frac{\cosh\left[\left(\frac{L_\tau}{2} - t\right) m_\pi\right]}{2 \sinh\left[\frac{L_\tau}{2} m_\pi\right]}}_{\approx \frac{e^{-m_\pi t}}{2}, \; t \ll \frac{L_\tau}{2}} , \quad m_\pi = \sqrt{\frac{2 \Sigma m}{F^2}}$$

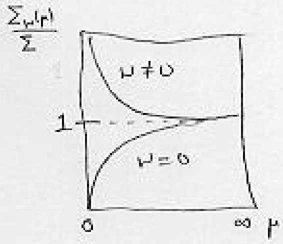
Question: If we have to carry out lattice simulations anyway, in order to determine F, Σ, what do we win over determining m_π directly with lattice methods?

Answer: We do not need to go to infinite volume in order to match with the predictions of the effective theory, since the effective theory correctly describes also the approach to the infinite-volume limit! The only requirement is that the box size L is large compared with the QCD-scale.

Recall: $\lambda_c^{(\text{glueball})} \approx 0.7\ \text{fm} \Rightarrow \underline{L \gtrsim 2.0\ \text{fm}}$.

(rather than $L \gg 9.0\ \text{fm}$ as on p. 118)

An example: Gauge field configurations can be classified according to an integer, the "topological charge ν." At a small volume, the effective theory predicts that



$$\Sigma_\nu(\mu) \equiv -\frac{1}{V}\left\langle \int d^4x\, \bar{u}u \right\rangle \approx \Sigma \cdot \begin{cases} \frac{|\nu|}{\mu} + \frac{\mu}{2|\nu|} + \mathcal{O}(\mu^3), & \nu \neq 0 \\ \left(\frac{1}{2} - \gamma_E + \ln\frac{2}{\mu}\right)\mu + \mathcal{O}(\mu^3), & \nu = 0 \end{cases}$$

Here γ_E = Euler constant ≈ 0.577, and $\mu = m\Sigma V = \frac{1}{2} m_\pi^2 F^2 V \lesssim 1$ here.

Monitoring dependence on ν, m, V, allows to determine Σ!

Results for the "quenched" theory $\left(\text{Det}[\not{D}] \equiv 1 \right)$:

$$F \simeq (105 \pm 5)\ \text{MeV}$$

$$\Sigma \simeq \left[(270 \pm 10)\ \text{MeV}\right]^3$$

$\Rightarrow$ promising, but have to be redone with the "unquenched" full QCD.

Summary

* In systems with a scale hierarchy, the ideal way to proceed is to try to factorise the problem into parts, rather than doing everything at once.

* In QCD, say, lattice simulations can be used for determining the parameters F, Σ of the effective theory, while the effective theory can be used for determining the masses of lightest hadrons.

* This procedure leads to a good explanation of the spectrum observed experimentally, and an interesting parametric dependence on the quark masses.

* Can the effective theory also be used to predict the masses of p, n ?? Maybe not ($m_p, m_n \sim m_{glueball}$, and are thus not light enough), although there are some intriguing possibilities (like "skyrmions", soliton-like extended objects.)

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