QCD: Goldstone phenomenon, low-energy effective theories

The Goldstone theorem [J. Goldstone, A. Salam, S. Weinberg, Phys. Rev. 127 (1962) 965]
states that if a continuous global symmetry gets broken, then there are massless particles in the spectrum. The number of massless particles depends on the symmetry breaking pattern.

The idea:

Consider, for simplicity, a single group. Let an element be $V_G = \exp(i \sum_{a=1}^{M} \omega^a T^a)$.
Denote the fields of the theory by ϕ. Here we take ϕ bosonic ($\sim \bar{q}q$?).

Action is symmetric $\Rightarrow$ $S[\phi] = S[V_G \phi]$.

Spontaneous symmetry breaking: $\phi_0 = \lim_{h \to 0} \lim_{V \to \infty} \frac{1}{V} \langle \int d^dx\, \phi(x) \rangle \neq 0$.

The direction of ϕ_0 (which is a vector) can be chosen at will, through h.

What is $V_G \phi_0$?

"Unbroken generators": $T^a \phi_0 = 0 \Rightarrow \exp(i\omega^a T^a)\phi_0 = \phi_0 \Rightarrow$ "leave vacuum invariant".

We are free to choose these to correspond to the last M_{unbroken} indices a.

"Broken generators": $T^a \phi_0 \neq 0 \Rightarrow \exp(i\omega^a T^a)\phi_0 = \phi_0' \neq \phi_0 \Rightarrow$ "do not leave vacuum invariant".

There are $M_{\text{broken}} = M - M_{\text{unbroken}}$ such generators.

Now:

$$S[\phi_0'] = S[\exp(i \sum_{a=1}^{M_{\text{broken}}} \omega^a T^a)\phi_0] = S[\phi_0]$$

$\Rightarrow$ there are M_{broken} "coordinates" ω^a around ϕ_0' which leave the action invariant, or "do not cost any energy".

$\Rightarrow$ there are M_{broken} exactly massless scalar degrees of freedom!

Let us apply this to chiral symmetry. Originally, $SU_L(N_f) \times SU_R(N_f) \Rightarrow 2(N_f^2-1)$ generators.

Unbroken ones: $SU_V(N_f) \Rightarrow N_f^2-1$

$\Rightarrow N_f^2-1$ broken generators

$\Rightarrow N_f^2-1$ massless Goldstone bosons! We denote them by w^a, $a=1,\dots,N_f^2-1$.

Goldstone manifold

Since we have chosen ϕ_0 to represent the ground state, we expect deviations from it, i.e. Goldstone modes, to be small in some sense. Without any loss of generality, we can thus parameterise them with a matrix:

$$U = \mathbb{1}_{N_f \times N_f} + i w^a T^a + O(w^2) \equiv \exp\left(\sum_{a=1}^{N_f^2-1} i w^a T^a\right) \in SU(N_f).$$

This space can be called the Goldstone manifold.

$\Longrightarrow$ How does U change in $V_L \in SU_L(N_f)$, $V_R \in SU_R(N_f)$?

* has to change somehow.
* simplest non-trivial change: linear.
* the "ground state" $U = \mathbb{1}$ has to be invariant, if and only if $V_L = V_R$.

$\Rightarrow$ Let us assign $U \to U' = V_L U V_R^{+}$.

Low-energy effective theory

In principle, the Goldstone modes are massless (or almost massless, if $0 < m \ll$ QCD-scale). Such light particles can usually be described by an effective low-energy theory (like in QCD itself we can often forget about c, b, t quarks, since they are heavy compared with the QCD-scale).

$\Rightarrow$ Write down the most general theory respecting spacetime and chiral symmetry! (Goldstone modes are physical $\Rightarrow$ gauge invariant $\Rightarrow$ no gauge symmetry!)

Simplest term: $S^{(0)} = \int d^4x \, \frac{F^2}{4} \, \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^+\right]$; F^2 = new parameter = "pion decay constant" $\propto$ (QCD-scale)^2 !

More complicated: $S^{(1)} = \int d^4x \; L \; \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^+\right] \mathrm{Tr}\left[\partial_\nu U \partial_\nu U^+\right]$, etc.

$\partial_\mu U \sim i T^a \partial_\mu w^a$ = "small", i.e. $\ll$ QCD-scale

$\Rightarrow S^{(1)} \ll S^{(0)}$!

Even though $M \neq 0$ breaks chiral symmetry, we can, most remarkably, also include $M \neq 0$ in the low-energy effective theory! The trick is to use the observation from p. 109: if $M \to V_L M V_R^+$, $M^+ \to V_R M^+ V_L^+$, then the action is formally chirally symmetric even for $M \neq 0$.

Simplest term: $S^{(2)} = \int d^4x \left\{ -\frac{\Sigma}{2} \, \mathrm{Tr}\left[U M^+ + M U^+\right] \right\}$; Σ = new parameter which, as we will presently see, equals the chiral condensate of p. 108.

More complicated: $S^{(3)} = \int d^4x \; K \; \mathrm{Tr}\left[U M^+ + M U^+\right] \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^+\right]$, etc.

$\Rightarrow S^{(3)} \ll S^{(2)}$.

Let us now look at some basic properties of the effective theory.

(A.) Order parameter in QCD:

$$\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\left\langle\int_x \bar\psi_1\psi_1\right\rangle = -\frac{1}{V}\frac{\partial}{\partial m_u}\ln\int \mathcal{D}A_\mu^a\mathcal{D}\psi\mathcal{D}\bar\psi\,\exp\left[-S^{(gauge)}-\int_x\bar\psi[\not{D}+M]\psi\right]\Bigg|_{M=0}$$

$\Rightarrow$ order parameter in effective theory:

$$= -\frac{1}{V}\frac{\partial}{\partial m_u}\ln\int\mathcal{D}U\exp\left[-\int_x\left(\cdots-\frac{\Sigma}{2}\mathrm{Tr}\left[UM^\dagger+MU^\dagger\right]\right)\right]\Bigg|_{M=0}$$

$U\approx\mathbb{1}$

$$= -\Sigma$$

OK!

(B.) Goldstone boson $\equiv$ pion field in effective theory:

$$U=e^{i\omega^aT^a}\quad\Rightarrow 2i\omega^aT^a\approx U-U^\dagger$$

$\Rightarrow$ let us put $M=iT^b$, $M^\dagger=-iT^b$

Then $\exp\left[-\int_x\left(\cdots-\frac{\Sigma}{2}\mathrm{Tr}[UM^\dagger+MU^\dagger]\right)\right]\approx\exp\left[\int_x\frac{\Sigma}{2}i\,\mathrm{Tr}\left[-UT^b+T^bU^\dagger\right]\right]$

$$\approx\exp\left[\int_x -i\frac{\Sigma}{2}\mathrm{Tr}\,T^b(U-U^\dagger)\right]$$

$$=\exp\left[\Sigma\int_x\frac{\omega^b}{2}\right]$$

$$\Rightarrow\omega^b=\frac{2}{\Sigma}\frac{\delta}{\delta\varepsilon(x)}\ln\int\mathcal{D}U\exp\left[-\int\left(\cdots\right)\right]\Bigg|_{M=\varepsilon(x)iT^b}$$

$\Rightarrow$ pion field in QCD:

$$\omega^b=\frac{2}{\Sigma}\frac{\delta}{\delta\varepsilon(x)}\ln\int\mathcal{D}A_\mu^a\mathcal{D}\psi\mathcal{D}\bar\psi\left[\cdots-\int_x\left(\bar\psi_LM\psi_R+\bar\psi_RM^\dagger\psi_L\right)\right]$$

$$=\frac{2}{\Sigma}\frac{\delta}{\delta\varepsilon(x)}\ln\int\mathcal{D}A_\mu^a\mathcal{D}\psi\mathcal{D}\bar\psi\left[\cdots-\int_x\left(\bar\psi_L iT^b\psi_R-\bar\psi_R iT^b\psi_L\right)\varepsilon(x)\right]$$

$$=\frac{2}{\Sigma}\frac{\delta}{\delta\varepsilon(x)}\ln\int\mathcal{D}A_\mu^a\mathcal{D}\psi\mathcal{D}\bar\psi\left[\cdots-\int_x\bar\psi\, iT^b\underbrace{(P_R-P_L)}_{\gamma_5}\psi\,\varepsilon(x)\right]$$

$$\approx-\frac{2}{\Sigma}\cdot\;\bar\psi_i(x)\,T^b_{ij}\,i\gamma_5\,\psi_j(x)$$

$T^b=\frac{1}{2}\begin{pmatrix}1&\\&-1\end{pmatrix}\Rightarrow\pi^0\sim u\bar u-d\bar d$

OK!

Summary:

- $S^{QCD} = S^{(gauge)} + \int d^4x \left\{ \bar{\Psi} \not{D} \Psi + \bar{\Psi}_L M \Psi_R + \bar{\Psi}_R M^\dagger \Psi_L \right\}$.

- Chiral symmetry broken explicitly for $M \neq 0$, spontaneously for $M = 0$

  $\Longrightarrow$ approximately massless Goldstone bosons for $M \neq 0$, exactly massless for $M = 0$!

- In both cases, their dynamics is described by an effective theory:

$$S^{eff} = \int d^4x \left\{ \frac{F^2}{4} \mathrm{Tr}\left[\partial_\mu U \partial_\mu U^\dagger\right] - \frac{\Sigma}{2} \mathrm{Tr}\left[U M^\dagger + M U^\dagger\right] + \dots \right\}, \quad U \in SU(N_f) .$$

- F^2 = "pion decay constant"; Σ = "chiral condensate" $\Big\}$ yet more manifestations of the QCD-scale of a few hundred MeV !

- pion = (pseudo-) Goldstone boson $\sim \bar{\Psi}_i T^b_{ij} i\gamma_5 \Psi_j$.

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