QCD: Global symmetries, symmetry breaking pattern

Formal continuum action: ⇒ **Lattice:**

$$S = \int d^4x \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi} [\gamma_\mu D_\mu + M] \psi \right\};$$

$$S = S^{(gauge)} + S^{(fermion)}$$

$\psi = \psi_{iA\alpha}(x)$ — i: flavour, A: colour, α: spinor, (x): spacetime

* **Space-time symmetries** ⇒ Translational and rotational symmetries clearly are broken explicitly by the lattice structure. The symmetries are, however, restored in the continuum limit. (Unless the ground state has a physical lattice structure, like crystal...)

Poincaré - transformations +
C = charge conjugation
P = parity reflection
T = time reflection.
Operate on spacetime (x) and spinor (α) indices (in Minkowski space)

* **Gauge symmetry** ⇒ Kept exactly by the lattice formulation. (This was the basic principle in constructing the lattice action.)

$SU(N_c)$; operates on colour (A) indices.

* **Flavour (or ~ chiral) symmetry** ⇒

Operates on flavour (i) indices.

- * broken explicitely with Wilson fermions (p. 92-93), but restored in the naive continuum limit.
- * respected exactly with Ginsparg-Wilson fermions.
- * But broken spontaneously!

⇒ For the moment, let us assume the use of Ginsparg-Wilson fermions, and use thus continuum notation in the flavour sector, to discuss the spontaneous symmetry breaking pattern.

Showing explicitely only flavour indices, the structure of the action is

$$S^{(\text{fermion})} = \int d^4x\ \bar{\psi}_i(x) \left[\delta_{ij} \not{D} + M_{ij} \right] \psi_j(x)$$

As on p. 92-93, we introduce a matrix $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma_5^\dagger = \gamma_5$, $\{\gamma_5, \gamma_\mu\} = 0$, $\gamma_5^2 = \mathbb{1}$, and define

$$P_R \equiv \frac{\mathbb{1}+\gamma_5}{2} \quad , \quad P_L \equiv \frac{\mathbb{1}-\gamma_5}{2} \quad .$$

These are projection operators:

$$P_R^2 = \frac{1}{4}\left[\mathbb{1} + 2\gamma_5 + \gamma_5^2\right] = P_R \qquad P_R\gamma_\mu = \gamma_\mu P_L$$

$$P_L^2 = P_L \qquad P_L\gamma_\mu = \gamma_\mu P_R$$

$$P_R P_L = \frac{1}{4}\left[\mathbb{1} - \gamma_5^2\right] = 0 \qquad \gamma_5 P_R = P_R$$

$$P_R + P_L = \mathbb{1} \ ; \qquad \gamma_5 P_L = -P_L \ .$$

The action can be written as

$$S^{(\text{fermion})} = \int d^4x\ \bar{\psi}_i(x) \left[\delta_{ij} \not{D} + M_{ij} \right] (P_R + P_L)\, \psi_j(x)$$

$$= \quad \text{---}\,''\,\text{---} \quad (P_R^2 + P_L^2) \quad \text{---}\,''\,\text{---}$$

$$= \int d^4x \left\{ \bar{\psi}_i P_L\, \delta_{ij} \not{D}\, P_R \psi_j + \bar{\psi}_i P_R\, \delta_{ij} \not{D}\, P_L \psi_j + \bar{\psi}_i P_R\, M_{ij}\, P_R \psi_j + \bar{\psi}_i P_L\, M_{ij}\, P_L \psi_j \right\}$$

Denote $\psi_{Li} \equiv P_L \psi_i$; $\bar{\psi}_i P_L = \psi_i^\dagger \gamma_0 P_L = \psi_i^\dagger P_R \gamma_0 = \psi_{Ri}^\dagger \gamma_0 \equiv \bar{\psi}_{Ri}$

$\psi_{Ri} \equiv P_R \psi_i$; $\bar{\psi}_i P_R = \bar{\psi}_{Li}$

$$\Rightarrow \quad S^{(\text{fermion})} = \int d^4x \left\{ \bar{\psi}_{Ri}\, \delta_{ij} \not{D}\, \psi_{Rj} + \bar{\psi}_{Li}\, \delta_{ij} \not{D}\, \psi_{Lj} + \bar{\psi}_{Li} M_{ij} \psi_{Rj} + \bar{\psi}_{Ri} M_{ij} \psi_{Lj} \right\}$$

Global flavour transformations:

$$\psi_{Li} \to \psi'_{Li} = [V_L]_{ij}\, \psi_{Lj} \quad , \quad V_L \in U(N_f) \equiv U_L(N_f)$$

$$\psi_{Ri} \to \psi'_{Ri} = [V_R]_{ij}\, \psi_{Rj} \quad , \quad V_R \in U(N_f) \equiv U_R(N_f) \ .$$

$\Rightarrow$ for $M = 0$, $S^{(\text{fermion})}$ is invariant under $\underline{U_L(N_f) \times U_R(N_f)}$!

A unitary matrix can be written $V = V' e^{i\alpha}$, where $V' = \frac{1}{(\det V)^{\frac{1}{N_f}}} V \in SU(N_f)$,
and $\alpha = \frac{1}{iN_f} \ln \det V \in \mathbb{R}$. (Recall that $V^\dagger V = \mathbb{1} \Rightarrow |\det V| = 1$).

Thus, $U(N_f) = SU(N_f) \times U(1)$.

The Abelian parts from here are usually combined into new ones:

$$\psi_i' = \left[e^{i\alpha_L} P_L + e^{i\alpha_R} P_R \right] \psi_i \equiv \left[e^{i(\alpha_V - \alpha_A)} P_L + e^{i(\alpha_V + \alpha_A)} P_R \right] \psi_i$$

$$\overset{\gamma_5 P_R = P_R,\ \gamma_5 P_L = -P_L}{=} e^{i\alpha_V} e^{i\alpha_A \gamma_5} (P_L + P_R) \psi_i = e^{i\alpha_V} e^{i\alpha_A \gamma_5} \psi_i$$

$\Rightarrow$ In the massless limit, the symmetry group is $\underline{SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1)}$.

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Spontaneous breaking of chiral symmetry

Like symmetries in $O(N)$-models, this one is also spont. broken at low temperatures!

In $O(N)$-models: (p 7) $\lim_{h \to 0} \lim_{N_{tot} \to \infty} \frac{1}{N_{tot}} \langle \vec{M} \rangle \cdot \frac{\vec{h}}{|\vec{h}|} \neq 0$.

In QCD: $\lim_{M \to 0} \lim_{V \to \infty} \frac{1}{V} \langle \int_x \bar{\psi}_1 \psi_1 \rangle \equiv -\Sigma \neq 0$.

Degenerate mass matrix, $M = \mathrm{diag}_{N_f \times N_f}(m, m, m, \ldots)$: $\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \cdot \frac{1}{N_f} \left\langle \int_x \sum_{i=1}^{N_f} \bar{\psi}_i \psi_i \right\rangle = -\Sigma \neq 0$.

This fact can be understood from phenomenology $\equiv$ hadron spectrum (see below) and by direct measurements on the lattice (see below), but there is (like for confinement) no satisfactory theoretical $\equiv$ analytic understanding of it, starting from the QCD action!

$\Sigma \equiv$ chiral condensate.

Symmetry breaking pattern:

$\frac{1}{N_f} \langle \sum_{i=1}^{N_f} \bar{\psi}_i \psi_i \rangle$

- is invariant under $U_V(1)$.
- is <u>not</u> invariant under $U_A(1)$: $\bar{\psi}_i \psi_i \to \bar{\psi}_i e^{2i\alpha_A \gamma_5} \psi_i$.

[It turns out, however, that the $U_A(1)$ symmetry is also broken explicitly by an <u>anomaly</u>, following from the functional integration measure.]

- is <u>not</u> invariant under $SU_L(N_f)$ or $SU_R(N_f)$, but <u>is</u> invariant under "$SU_V(N_f)$", i.e. the case that $V_L = V_R \equiv V$!

$\left(\sum_i \bar{\psi}_i \psi_i = \sum_i \{ \bar{\psi}_{Li} \psi_{Ri} + \bar{\psi}_{Ri} \psi_{Li} \} \right)$

Thus we say that chiral symmetry breaks as

$$SU_L(N_f) \times SU_R(N_f) \times U_V(1) \longrightarrow SU_V(N_f) \times U_V(1) .$$

A useful trick:

We have seen that the mass matrix M breaks the chiral symmetry explicitly. We can, however, learn something about the way in which this happens by writing ($M = \mathrm{diag}(m_u, m_d, m_s, \ldots)$)

$$\bar{\psi}_{Li} M_{ij} \psi_{Rj} + \bar{\psi}_{Ri} M_{ij} \psi_{Lj} \longrightarrow \bar{\psi}_{Li} M_{ij} \psi_{Rj} + \bar{\psi}_{Ri} M^{+}_{ij} \psi_{Lj} ,$$

and <u>assigning</u> a transformation law $\begin{cases} M \to V_L M V_R^{+} \\ M^{+} \to V_R M^{+} V_L^{+} \end{cases}$ to M.

Then $S^{(\text{fermion})}$ <u>is</u> formally invariant in $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$ even for $M \neq 0$. This will be useful soon.

Summary:

- In the "chiral limit" $\frac{M}{\text{QCD-scale}} \longrightarrow 0$, the theory has a global flavour symmetry $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$. An additional Abelian symmetry, $U_A(1)$, has been broken by a quantum-mechanical anomaly (this is a remarkable story of its own).

- For low temperatures, the symmetry is spontaneously broken.

- Order parameter = chiral condensate = $\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \frac{1}{N_f} \left\langle \int_x \sum_i \bar{\psi}_i \psi_i \right\rangle$.

- The part of the symmetry that remain unbroken:

$$SU_V(N_f) \times U_V(1).$$