Properties of pure gauge theory; "QCD scale"

Hadron spectrum (from "Particle Data Group", pdg.lbl.gov)

	Particle:	Mass:	Quark assignment:	
Mesons	π^0	135 MeV	$u\bar{u} - d\bar{d}$	
	$\pi^\pm$	140 MeV	$u\bar{d}, d\bar{u}$	$\Rightarrow m_u, m_d \lesssim 70$ MeV
	η	547 MeV	$\sim u\bar{u} + d\bar{d} - 2s\bar{s}$	$\Rightarrow m_s \lesssim 250$ MeV
	ρ	771 MeV	$u\bar{u} - d\bar{d}$	
	ω	783 MeV	$\sim u\bar{u} + d\bar{d} + s\bar{s}$	
	η'	958 MeV	$u\bar{u} + d\bar{d} + s\bar{s}$	
	$\vdots$			
Strange mesons	$K^0, \bar{K}^0$	~ 498 MeV	$d\bar{s}, \bar{s}d$	
	$K^\pm$	494 MeV	$u\bar{s}, \bar{s}u$	
Charmed mesons	$D^0, \bar{D}^0$	~ 1864 MeV	$u\bar{c}, \bar{c}u$	$\Rightarrow m_c \lesssim 1800$ MeV
	$D^\pm$	1869 MeV	$d\bar{c}, \bar{c}d$	
$c\bar{c}$ mesons	J/Ψ	3097 MeV	$c\bar{c}$	
Baryons	p	938 MeV	uud	
	n	940 MeV	udd	
	$\vdots$			
Exotics	pentaquark, Θ^+	1540 MeV	$udud\bar{s}$	
$\vdots$				

All of this needs to be explained!
On the lattice, we are free to tune the quark masses at will.
What happens to the spectrum then?
In particular, if all quarks are made infinitely heavy ($m_u, m_d, m_s, \ldots \to \infty$), what kind of states are left over?

Glueball $\equiv ggg\cdots$ **What is its mass?**

[In QED, purely photonic states are exactly massless!]

In the limit $m_{u,\dots} \to \infty$, quarks are too heavy to propagate (they "decouple"), and the question can be answered within pure gauge theory,

$$S^{(\text{gauge})} = \frac{1}{g_0^2} \sum_{\tilde{x}} \sum_{\mu,\nu} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(\tilde{x}) \right], \qquad P_{\mu\nu} = \square .$$

This theory has a most remarkable property: it has only one dimensionless parameter (and $g_0^2 \to 0$ in the continuum limit), but predicts the existence of very non-trivial bound states, with non-zero masses! This is called "dimensional transmutation": a mass scale is generated almost out of nothing. The absolute values of masses are not predictable, but their ratios are.

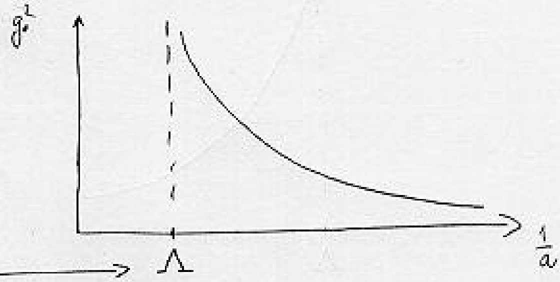
The simplest way to understand the generation of a mass scale: renormalisation group equation.

p. 75 : $$a \frac{\partial}{\partial a} g_0^2 \Big|_{g_R^2} = + \frac{11 N_c}{24 \pi^2} g_0^4 .$$

Ansatz: $$g_0^2 = \frac{c}{\ln(a\Lambda)} \quad ; \quad \frac{\partial}{\partial a} g_0^2 = - \frac{c}{\ln^2(a\Lambda)} \cdot \frac{1}{a}$$

$$\Rightarrow \quad -\frac{1}{c} \cdot g_0^4 = + \frac{11 N_c}{24\pi^2} g_0^4 \quad \Rightarrow \quad c = - \frac{24\pi^2}{11 N_c}$$

$$\Rightarrow \quad g_0^2(a) \Big|_{g_R^2} = - \frac{24\pi^2}{11 N_c \ln(a\Lambda)} = \frac{24\pi^2}{11 N_c \ln\left(\frac{a^{-1}}{\Lambda}\right)}$$



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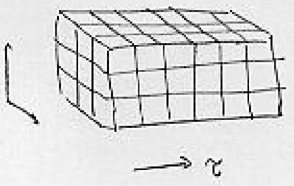
"QCD-scale" has been generated! One can exchange $g_0^2 \longleftrightarrow \Lambda$!

But note that this is specific to the lattice background, or more generally, to the regularisation scheme used. So this particular number is not directly physical, but the fact that some scale gets generated, is.

Let us rather directly measure the lowest mass in the system!

Recall from p. 65: physical masses can be determined from the exponential falloff of correlation functions in Euclidean space:

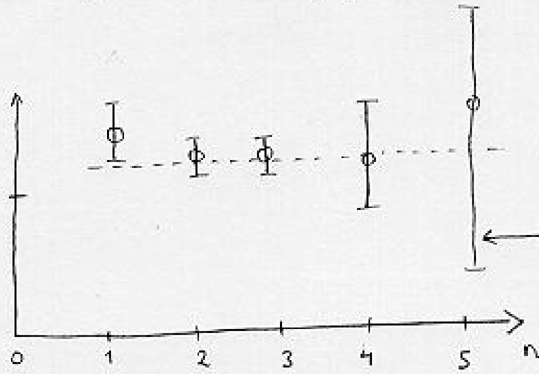
spatial direction



$$C(\tau) \equiv \langle O(\tau)\, O(0)\rangle - \langle O(\tau)\rangle\langle O(0)\rangle \sim e^{-m\tau}$$

Since τ is in lattice units ($\tau = a\cdot n$, $n \in \mathbb{N}$), masses will be in lattice units:

$$ma = \ln\left[\frac{C(\tau)}{C(\tau+a)}\right]$$



at large distances error bars explode because $C(\tau)$ is exponentially small.

The simplest operators: e.g. $O(\tau) = \sum_{\bar{x}} a^3\, P_{12}(\tau, \bar{x})$

The spatial volume and $S^{(gauge)}$ have certain symmetry groups: rotations, parity transformation $\bar{x} \to -\bar{x}$, charge conjugation.
Operators can be decomposed into parts transforming under various irreducible representations of these groups, characterised by J^{PC}.

One finds, e.g.:

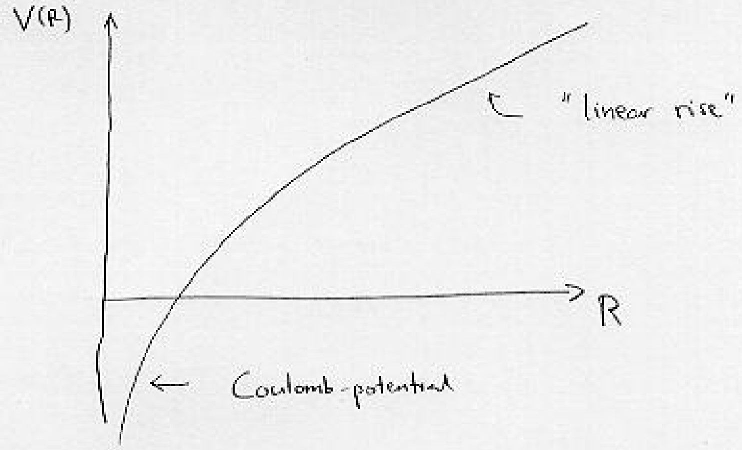
[M.J. Teper, C. Michael, Nucl. Phys. B 314 (89) 347, etc]

J^{PC}	$\beta_G = \frac{2N_c}{g^2} = 6.0$	$\beta_G = \frac{2N_c}{g^2} = 6.2$	$\rightarrow$ closer to continuum
0^{++}	0.710 ± 0.022	0.560 ± 0.020	
2^{++}	1.11 ± 0.03	0.845 ± 0.020	
0^{-+}	1.25 ± 0.14	0.92 ± 0.08	
2^{-+}	1.28 ± 0.15	1.20 ± 0.09	
1^{+-}	1.33 ± 0.14	0.984 ± 0.045	

Fixing the scale through a non-relativistic quark model :

More information can be obtained by considering quarks which are heavy but not infinitely heavy: e.g., J/Ψ, or various $b\bar{b}$ states, Υ's.

They can be described (approximatively) by first assuming $m_b = \infty$, and determining the static potential $V(R)$ in pure gauge theory (p.77) :



And then solving a non-relativistic Schrödinger equation :

$$\left[-\frac{\hbar^2}{m_b}\vec{\nabla}^2 + V(R) \right] \Psi_{n\ell m} = E_{n\ell m} \Psi_{n\ell m}$$

Matching with several masses, produces an independent conversion from a given β_G to physics.

Examples :

$$\beta_G = 6.0 \Rightarrow a = \frac{r_0}{5.37}, \quad r_0 = 0.5\,\text{fm}, \quad \text{fm} = \text{fermi} = 10^{-15}\,\text{m}$$

$$\beta_G = 6.2 \Rightarrow a = \frac{r_0}{7.37} \quad \ldots$$

Master relation : $\text{GeV} \cdot \text{fm} = 5.07$ in units where $\hbar c = 1$! $\left(\hbar = \frac{h}{2\pi}\right)$

$\Big[$ proof: $\text{GeV fm} = 10^9 \cdot \text{eV} \cdot \frac{1}{\hbar c} \cdot 10^{-15}\,\text{m} = 10^9 \cdot 1.602 \cdot 10^{-19}\,\text{J} \cdot \frac{1}{1.054 \cdot 10^{-34}\,\text{Js}} \cdot \frac{1}{2.9979 \cdot 10^8 \frac{\text{m}}{\text{s}}} \cdot 10^{-15}\,\text{m}$ $\Big]$

$$\Rightarrow m(0^{++}) = 0.560 \cdot \frac{1}{a}\Big|_{\beta_G = 6.2} = 0.560 \cdot \frac{7.37}{0.5\,\text{fm}} = 0.560 \cdot \frac{7.37}{0.5} \cdot \frac{1}{5.07} \cdot \text{GeV}$$

$$= \underline{1630\ \text{MeV}} \quad !$$

(a careful extrapolation from several β_G brings this to ≈ 1730 MeV)

Summary:

- Pure gauge theory has no tunable parameters in the continuum limit $g_0^2 \to 0$.
- Yet it has a rich spectrum of massive bound states, glueballs.
- The overall mass scale cannot be predicted, but has to be fit to experimental data, using e.g. heavy quark bound states and the static potential. This characteristic scale is called the "QCD scale".
- Once the scale is fixed, everything else is predicted.
- Examples of observables which can be used to characterise the QCD scale:

  * lightest glueball mass, $m(0^{++}) \approx 1700$ MeV
  * "string tension σ" $\equiv$ coefficient of linear rise in static potential at large R: $\sqrt{\sigma} \approx 470$ MeV
  * $\Lambda_{\overline{MS}}$ = scale parameter related to a certain continuum regularization scheme: $\Lambda_{\overline{MS}} \approx 250$ MeV

  ⋮

- Quarks can now be divided into light and heavy flavours:

$$m_u, m_d, m_s \ll \text{QCD scale}$$
$$m_c, m_b, m_t \gg \text{QCD scale}.$$

Often a reasonable first approximation is $m_u = m_d = m_s = 0$, $m_c = m_b = m_t = \infty$!

The methods that apply for light and heavy flavours are completely different.

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