Overview of contemporary direction in lattice gauge field theories

Parts of the Standard Model of particle physics:

Theory	Expansion parameter of weak coupling expansion
* electromagnetic interactions: QED [U(1) + fermion]	$\alpha_{EM} = \frac{e^2}{4\pi\epsilon_0\hbar c} \overset{\text{"particle physics units"}}{=} \frac{e^2}{4\pi} \approx \frac{1}{137} \ll 1$. $\Rightarrow$ perturbative computations are accurate with up to ~10 digits.
* Weak interactions: the Weinberg-Salam model [SU(2) + Higgs + fermions]	$\alpha_W = \frac{g_w^2}{4\pi} \approx \frac{1}{30}$. $\Rightarrow$ perturbative computations are still accurate at the % level.
* Strong interactions: Quantum Chromodynamics, QCD [SU(3) + quarks]	$\alpha_S = \frac{g_s^2}{4\pi} \approx 0.3 \ldots 1.0 \ldots$ large. $\Rightarrow$ perturbative computations in terms of the renormalised coupling are in general not reliable for physical observables. $\Rightarrow$ **QCD is the prime application of "non-perturbative" lattice techniques.**
* gravitational interactions: no quantum theory agreed upon yet. One possibility is string theory — might be perturbative or non-perturbative.	

$\Rightarrow$ **In the following, concentrate on QCD.**

Lattice QCD action in explicit component form

$$S^{(\text{fermion})} = \sum_{\bar{x},\bar{y}} a^4 \sum_{i,j=1}^{N_f} \sum_{A,B=1}^{N_c} \sum_{\alpha,\beta=1}^{4} \overline{\Psi}_{iA\alpha}(\bar{x}) \left\{ \not{D}_{ij;AB;\alpha\beta}(\bar{x},\bar{y}) + M_{ij}\,\delta_{AB}\,\delta_{\alpha\beta}\,\delta_{\bar{x}\bar{y}} \right\} \Psi_{jB\beta}(\bar{y})$$

i,j = flavours ; N_f = 2 (u,d)
3 (u,d,s)
4 (u,d,s,c)
5 (u,d,s,c,b)
6 (u,d,s,c,b,t)

A,B = colours ; $N_c = 3$

α,β = spinor indices

$\not{D}_{ij;AB;\alpha\beta}(\bar{x},\bar{y})$ = Dirac operator (often denoted also just by D)

Wilson Dirac operator : p. 93

$$\not{D}_{ij;AB;\alpha\beta}(\bar{x},\bar{y}) = \delta_{ij} \left\{ \frac{4r}{a}\,\delta_{\alpha\beta}\,\delta_{AB}\,\delta_{\bar{x}\bar{y}} - \frac{1}{2a}\sum_{\mu=0}^{3} \Big[\left(r\,\delta_{\alpha\beta} + [\gamma_\mu]_{\alpha\beta}\right)\left[U_\mu^\dagger(\bar{y})\right]_{AB}\,\delta_{\bar{x},\bar{y}+a\hat{\mu}} + \left(r\,\delta_{\alpha\beta} - [\gamma_\mu]_{\alpha\beta}\right)\left[U_\mu(\bar{x})\right]_{AB}\,\delta_{\bar{y},\bar{x}+a\hat{\mu}} \Big] \right\}$$

M_{ij} = mass matrix = $\mathrm{diag}_{N_f\times N_f}(m_u, m_d, m_s, \dots)$

Bare parameters : $\Leftrightarrow$ **Renormalized parameters** (many possible definitions)

Bare parameters	Renormalized parameters
g_0^2	g_R^2
$m_{u,0}$	$m_{u,R}$
$m_{d,0}$	$m_{d,R}$
$m_{s,0}$	$m_{s,R}$
$m_{c,0}$	$m_{c,R}$
$m_{b,0}$	$m_{b,R}$
$m_{t,0}$	$m_{t,R}$

Short-hand notation : $S^{(\text{fermion})} = \sum_{\bar{x}} a^4\, \overline{\Psi}\,(\not{D} + M)\,\Psi$.

After carrying out the Grassmann integral over the quark fields, which is usually written in the form ($\hbar = 1$)

$$Z^{(fermion)} = \int \left\{ \prod_{\bar{x}} \mathcal{D}\left(a^{3/2}\Psi(\bar{x})\right) \right\} \left\{ \prod_{\bar{x}} \mathcal{D}\left(a^{3/2}\bar{\Psi}(\bar{x})\right) \right\} \exp\left\{ - \sum_{\bar{x},\bar{y}} a^{3/2}\bar{\Psi}(\bar{x}) \left[a\not{D}(\bar{x},\bar{y}) + aM \right] a^{3/2}\Psi(\bar{y}) \right\}$$

$$\equiv \mathrm{Det}\left[a\not{D} + aM \right] = \prod_i \lambda_i$$

$$\frac{1}{Z^{(fermion)}} \int \left\{ \text{---''---} \right\} \left\{ \text{---''---} \right\} \Psi(\bar{x})\bar{\Psi}(\bar{y}) \exp\left\{ \text{---''---} \right\}$$

$$= \left[a\not{D} + aM \right]^{-1}(\bar{x},\bar{y}) = \underbrace{\sum_i}_{\text{spectral representation}} \frac{v_i(\bar{x})\, v_i^{\dagger}(\bar{y})}{\lambda_i} ,$$

where $\left[a\not{D} + aM \right] v_i(\bar{x}) = \lambda_i v_i(\bar{x})$, for each given configuration $U_\mu(\bar{x})$, what remains to be computed is

$$Z^{(full)} = \int \left\{ \prod_{\bar{x},\mu} \mathcal{D}U_\mu(\bar{x}) \right\} \mathrm{Det}\left[a\not{D} + aM \right] e^{-S^{(gauge)}}$$

$$\langle \mathcal{O} \rangle^{full} = \frac{1}{Z^{(full)}} \int \left\{ \prod_{\bar{x},\mu} \mathcal{D}U_\mu(\bar{x}) \right\} \mathrm{Det}\left[a\not{D} + aM \right] \mathcal{O}\left\{ \Psi\bar{\Psi} \to \left[a\not{D} + aM \right]^{-1} \right\} e^{-S^{(gauge)}}$$

So everything reduces to:

* generating configurations of $U_\mu(\bar{x})$ by Monte Carlo.

* solving for eigenvalues and eigenfunctions of the Dirac operator for such configurations.

A significant (ad hoc) simplification: "quenched approximation" $\Longleftrightarrow$ $\mathrm{Det}\left[a\not{D} + aM \right] \equiv 1$.

Challenges

"Technical":

- **Algorithms** ⇒ * How to compute efficiently (= fast, precisely) the eigenvalues and eigenfunctions of the $\left[4 \times N_c \times N_f \times \left\{\prod_{\mu=0}^{3} N_\mu\right\}\right] \times \left[4 \times N_c \times N_f \times \left\{\prod_{\mu=0}^{3} N_\mu\right\}\right]$-dimensional Dirac operator?
  - * How to compute the determinant ≡ "Unquenching".
  - * How in general to optimise the update; how to "parallelise"; etc.

  ≡ applied mathematics, computer science, computational physics.

- **Machines** ⇒ Special-purpose for QCD?

"Theoretical":

* How to find a discretised Dirac operator which respects all the continuum symmetries?

  ⇒ "Ginsparg-Wilson" fermions.

* How to approach the continuum limit as optimally / fast as possible?

  ⇒ "improved actions"

* What kind of observables to consider / measurements to carry out? I.e., asking the right questions: lattice allows to probe QCD in ways not possible in accelerator experiments, and thus learn things about it in unexpected ways.

* ...

"Phenomenological"

1. Is QCD really the correct theory of strong interactions? Quarks and gluons are not observed directly, but does the action written in terms of them still predict the properties of mesons, proton, neutron, ... ?

2. If QCD is correct, determine its parameters ($g_R^2, m_{i,R}$) by matching e.g. pion, proton, etc masses to observed values.

3. Is Weinberg-Salam the correct theory of weak interactions? Even though weak interactions are themselves perturbative, the initial or final states may be hadrons, which can only be described by QCD.

4. How do quarks and gluons behave at the extremely high temperatures relevant for the Early Universe? There are no experimental probes, lattice is the only way!

5. How do quarks and gluons behave at the extremely high densities relevant for the cores of e.g. neutron stars? Again no direct experimental probes, need lattice.

6. Can we understand the property of quark confinement? Although a full "understanding" would have to be analytic, lattice can be used for determining observables which are not directly measurable in experiment, and yet reflect the confining dynamics. Eg: static potential for $N_f = 0$.

⋮

In the following lectures we go through a few selected "phenomenological" topics.