Wilson fermions

Various solutions have been presented for the doubling problem. The simplest one is by K.G. Wilson, and we consider this one here.

[Some others:

- "Ginsparg-Wilson fermions": a general class, including "overlap" = "Neuberger", "domain wall", "perfect action", ... Theoretically superior, but extremely costly in practice.
- "staggered" = "Kogut-Susskind". Theoretically dubious, but inexpensive to simulate ⇒ frequently used (somewhat unfortunately).
- "twisted mass QCD" — somewhere in between.]

The idea of Wilson fermions:

Add a term to the action which vanishes in the naive continuum limit [so that it does nothing to the "physical" part of the phase space], but not in the unphysical corners, where it "lifts" the lattice momentum away from zero.

An example:

$$S = S^{(old)} - \overset{\text{real number chosen at will; for instance, } r=1.}{\frac{r}{2}} \sum_{\bar{x}} a^4 \cdot a \cdot \bar{\psi}(\bar{x}) \,\hat{\Box}\, \psi(\bar{x}), \qquad \text{where}$$

$$\hat{\Box}\,\psi(\bar{x}) \equiv \sum_{\mu} \frac{\psi(\bar{x}+a\hat{\mu}) - 2\psi(\bar{x}) + \psi(\bar{x}-a\hat{\mu})}{a^2}$$

In momentum space:

$$\int_p \bar{\psi}(p)\psi(p) \left\{ -\frac{r}{2} \cdot a \cdot \frac{1}{a^2} \cdot \sum_{\mu} \left[\overbrace{e^{iap_\mu} - 2 + e^{-iap_\mu}}^{(e^{\frac{iap_\mu}{2}} - e^{-\frac{iap_\mu}{2}})^2} \right] \right\}$$

$$= \int_p \bar{\psi}(p)\psi(p) \left\{ + \frac{ra}{2} \sum_{\mu} \tilde{p}_\mu^2 \right\}$$

Inverse propagator now : $i\mathring{\not{p}} + m + \frac{1}{2} a r \tilde{p}^2$

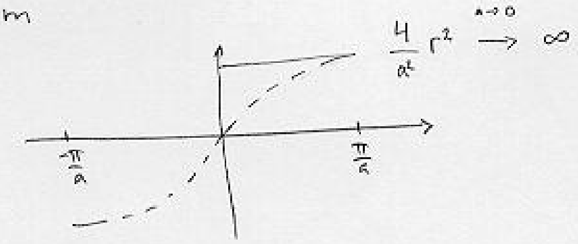
Propagator:

$$\langle \psi_\alpha(p)\, \bar{\psi}_\beta(q) \rangle = \delta(p-q)\, \frac{-i\mathring{\not{p}} + m + \frac{1}{2} a r \tilde{p}^2}{\mathring{p}^2 + \left(m + \frac{1}{2} a r \tilde{p}^2\right)^2}$$

Now the denominator reads :

$$\underbrace{\mathring{p}^2 + \frac{1}{4} a^2 r^2 \tilde{p}^4}_{} + \underbrace{m\left(m + a r \tilde{p}^2\right)}_{\geq m}$$

Along one axis :



So problem seems solved: the unphysical zeros have been lifted by a large amount. There is, however, a price to pay.

Chiral symmetry

Let us introduce $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$; $\gamma_5^+ = \gamma_5$; $\{\gamma_5, \gamma_\mu\} = 0$, $\mu = 0,1,2,3$.

The naive action $S = \sum_{\bar{x}} a^4\, \bar{\psi}(\bar{x}) \left\{ \frac{1}{2a} \sum_\mu \gamma_\mu \left[\psi(\bar{x}+a\hat{\mu}) - \psi(\bar{x}-a\hat{\mu}) \right] + m\, \psi(\bar{x}) \right\}$

has the global "vector" $U_V(1)$ symmetry

$$\psi \to \psi' = e^{i\alpha}\psi$$
$$\bar{\psi} \to \bar{\psi}' = e^{-i\alpha}\bar{\psi}$$

,

and, for $m \to 0$, an additional "axial" (or ~ "chiral") $U_A(1)$ symmetry

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi$$
$$\gamma_\mu\psi \to \gamma_\mu\psi' = e^{-i\alpha\gamma_5}\gamma_\mu\psi \qquad \left(\text{since } \{\gamma_5, \gamma_\mu\} = 0\right)$$
$$\psi^+ \to \psi'^+ = \psi^+ e^{-i\alpha\gamma_5}$$
$$\psi^+\gamma_0 \to \psi'^+\gamma_0 = \psi^+\gamma_0\, e^{+i\alpha\gamma_5} \qquad \left(\text{since } \{\gamma_5, \gamma_0\} = 0\right)$$

The axial symmetry is only broken by the mass term:

$$\bar{\Psi}\, m\, \Psi \;\rightarrow\; \bar{\Psi}\, m\, e^{2i\alpha\gamma_5}\, \Psi$$

It is thus recovered in the "chiral" limit $m \to 0$. This plays a very important role for the properties of QCD, since the up and down quark masses are "almost" zero.

The problem with the Wilson term is that it breaks the axial symmetry even when $m \equiv 0$!

$$a\bar{\Psi}\,\hat{\Box}\,\Psi \;\rightarrow\; a\bar{\Psi}\,\hat{\Box}\, e^{2i\alpha\gamma_5}\,\Psi$$

Formally the breaking vanishes in the naive continuum limit $a \to 0$, but this is not necessarily enough.

This problem is only solved by the "Ginsparg-Wilson" fermions...

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Adding gauge fields

Almost trivial: $\Psi \to g(\bar{x})\Psi$, $\bar{\Psi} \to \bar{\Psi}\, g^{+}(\bar{x})$.

$$S = \sum_{\bar{x}} a^4\, \bar{\Psi}(\bar{x}) \left\{ \left(m + 4\frac{r}{a}\right)\Psi(\bar{x}) + \frac{1}{2}\sum_{\mu}\left[\left(\frac{\gamma_\mu}{a} - \frac{r}{a}\right)\Psi(\bar{x}+a\hat{\mu}) - \left(\frac{\gamma_\mu}{a} + \frac{r}{a}\right)\Psi(\bar{x}-a\hat{\mu})\right]\right\}$$

$$\Longrightarrow \sum_{\bar{x}} a^4 \left\{ \bar{\Psi}(\bar{x})\left(m + \frac{4r}{a}\right)\Psi(\bar{x}) - \frac{1}{2a}\sum_{\mu}\Big[(r+\gamma_\mu)\, \bar{\Psi}(\bar{x}+a\hat{\mu})\, U_\mu^{+}(\bar{x})\, \Psi(\bar{x}) \right.$$
$$\left. + (r-\gamma_\mu)\, \bar{\Psi}(\bar{x})\, U_\mu(\bar{x})\, \Psi(\bar{x}+a\hat{\mu}) \Big] \right\}$$

How to proceed in practice?

Grassmann variables are a somewhat strange mathematical construction — there is surely no way to have them as variables in a computer, which only recognises real numbers. So we have to integrate them out!

We write: $S^{(\text{fermion})} = \sum_{\bar{x},\bar{y}} a^4 \,\bar{\psi}(\bar{x})\, D_{\bar{x}\bar{y}}\, \psi(\bar{y})$, wher $D_{\bar{x}\bar{y}} \equiv$ Dirac operator.

Then:

$$\langle \mathcal{O} \rangle = \frac{\int \left\{ \prod_{\bar{x},\mu} \mathcal{D}U_\mu(\bar{x})\, \mathcal{D}\psi(\bar{x})\, \mathcal{D}\bar{\psi}(\bar{x}) \right\} \mathcal{O}[U_\mu, \psi, \bar{\psi}]\, e^{-S^{(\text{gauge})}}\, e^{-S^{(\text{fermion})}}}{\int \left\{ \prod_{\bar{x},\mu} \mathcal{D}U_\mu(\bar{x})\, \mathcal{D}\psi(\bar{x})\, \mathcal{D}\bar{\psi}(\bar{x}) \right\} e^{-S^{(\text{gauge})}}\, e^{-S^{(\text{fermion})}}}$$

$$= \frac{\int \left\{ \prod_{\bar{x},\mu} \mathcal{D}U_\mu(\bar{x}) \right\} e^{-S^{(\text{gauge})}} \left[\int \left\{ \prod_{\bar{x}} \mathcal{D}\psi(\bar{x})\, \mathcal{D}\bar{\psi}(\bar{x}) \right\} \mathcal{O}[U_\mu, \psi, \bar{\psi}]\, e^{-S^{(\text{fermion})}} \right]}{\int \left\{ \prod_{\bar{x},\mu} \mathcal{D}U_\mu(\bar{x}) \right\} e^{-S^{(\text{gauge})}} \left[\int \left\{ \prod_{\bar{x}} \mathcal{D}\psi(\bar{x})\, \mathcal{D}\bar{\psi}(\bar{x}) \right\} e^{-S^{(\text{fermion})}} \right]}$$

These are "Gaussian" integrals in a given background of $U_\mu(\bar{x})$'s!

P. 89:

$$\langle 1 \rangle = \int \left\{ \prod_{\bar{x}} \mathcal{D}\psi(\bar{x})\, \mathcal{D}\bar{\psi}(\bar{x}) \right\} e^{-\sum_{\bar{x}\bar{y}} a^4 \bar{\psi}(\bar{x}) D_{\bar{x}\bar{y}} \psi(\bar{y})} = \mathrm{Det}\left[a^4 D_{\bar{x}\bar{y}} \right] = \prod_i \lambda_i \quad \text{(eigenvalues)},$$

$$\langle \psi(\bar{v}) \bar{\psi}(\bar{w}) \rangle = \int \left\{ \prod_{\bar{x}} \mathcal{D}\psi(\bar{x})\, \mathcal{D}\bar{\psi}(\bar{x}) \right\} \psi(\bar{v})\, \bar{\psi}(\bar{w})\, e^{-S^{(\text{fermion})}} = \langle 1 \rangle \cdot \frac{\langle \psi(\bar{v}) \bar{\psi}(\bar{w}) \rangle}{\langle 1 \rangle}$$

$$= \mathrm{Det}\left[a^4 D_{\bar{x}\bar{y}} \right] \cdot \left[a^4 D_{\bar{x}\bar{y}} \right]^{-1}_{\bar{v},\bar{w}}$$

The inverse can be solved for using the spectral representation:

$$M |v_i\rangle = \lambda_i |v_i\rangle \Rightarrow M^{-1} = \sum_i \frac{|v_i\rangle\langle v_i|}{\lambda_i}, \quad \text{if } \sum_i |v_i\rangle\langle v_i| = \mathbb{1}.$$

$\Rightarrow$

* The problem reduces to finding the eigenvalues and eigenvectors of the lattice Dirac operator in a given $U_\mu(\bar{x})$ background, and averaging then over backgrounds. (Expensive!)
* $\mathcal{O}[U_\mu, \psi, \bar{\psi}]$ reduces to propagators, using Wick's theorem; and each of the propagators is expressed in terms of eigenvalues.

Boundary conditions

To solve for the Dirac operator eigenfunctions in a finite box, we need to specify boundary conditions for the solutions.

(a) $T=0$, large volume

In the limit of a large four-volume, boundary conditions do not matter. One could use, for instance, periodic boundary conditions in all directions for the solutions, as we have been doing so far for bosonic fields.

(b) $T>0$

In the finite temperature case, the situation is different.

For bosons, we had

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int_{-\infty}^{\infty} d\phi \langle \phi | e^{-\beta \hat{H}} | \phi \rangle \Rightarrow \tau = 0 \dots \beta\hbar, \quad \phi(0,\bar{x}) = \phi(\beta\hbar, \bar{x}).$$

$\Rightarrow$ have to use periodic b.c.'s

For fermions, it turns out that the interval is again $\tau = 0 \dots \beta\hbar$, but now one has to impose anti-periodic boundary conditions!

$$\psi(\beta\hbar, \bar{x}) \equiv -\psi(0, \bar{x})$$
$$\bar{\psi}(\beta\hbar, \bar{x}) \equiv -\bar{\psi}(0, \bar{x})$$

This is related to Fermi-Dirac statistics / the Grassmann nature of variables.

Proof: a topic in finite-temperature field theory.