Fermions

This time we start from continuum, carry out a "naive discretisation" (the reverse of naive continuum limit), and then try to come back towards continuum, more honestly. And there will be a problem!

For the moment, $U_\mu(\vec{x}) \Rightarrow \mathbb{1}$ — $U_\mu \neq \mathbb{1}$ will be inserted at the end.

Let us first recall how the Euclidean continuum action for fermion looks like.

- Dirac equation: $(i\gamma^\mu\partial_\mu - m)\underbrace{\psi}_{\text{4-component vector, "spinor"}} = 0$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$

- Minkowskian action:

$$\exp\left(\frac{i}{\hbar} S_M\right),$$

$$S_M = \int dt \int d^3x \; \bar\psi \left[i\gamma^\mu\partial_\mu - m\right]\psi, \qquad \bar\psi = \psi^\dagger\gamma^0$$

$$\Rightarrow \frac{\delta S_M}{\delta \bar\psi} = 0 \quad \text{gives the Dirac equation.}$$

- Euclidean action: "Wick rotation" (p. 61) : $\underline{it \to \tau}$.

$$i\partial_t = -\partial_\tau, \quad i\,dt = d\tau$$

We define "Euclidean γ-matrices" by

$$\gamma_0^E \equiv \gamma^0, \quad \gamma_i^E \equiv -i\gamma^i$$

Then $\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}$, $(\gamma_\mu^E)^\dagger = \gamma_\mu^E$.

In the following, the superscript E will be dropped.

Thus: $i\gamma^\mu\partial_\mu - m = i\gamma^0\partial_t + i\gamma^i\partial_i - m$

$$\Rightarrow -\gamma_0\partial_\tau - \gamma_i\partial_i - m = -(\gamma_\mu\partial_\mu + m).$$

$$\exp\left(\frac{i}{\hbar}S_M\right) \Rightarrow \exp\left(-\frac{1}{\hbar}\int d\tau \int d^3x \; \bar\psi(\gamma_\mu\partial_\mu + m)\psi\right) \equiv \exp\left(-\frac{1}{\hbar}S_E\right),$$

$$\underline{S_E = \int d\tau \int d^3x \; \bar\psi(\gamma_\mu\partial_\mu + m)\psi}.$$

Naive discretisation

We now put the system on a lattice, as usual.

Attempt 1:

$$S = \sum_{\bar{x}} a^4 \, \bar{\psi}(\bar{x}) \left\{ \frac{1}{a} \gamma_\mu \left[\psi(\bar{x}+a\hat{\mu}) - \psi(\bar{x}) \right] + m \psi(\bar{x}) \right\}$$

The problem is that this is formally not "Hermitean", or "real":

$$\left\{ \psi^\dagger(\bar{x}) \gamma_0 \gamma_\mu \left[\psi(\bar{x}+a\hat{\mu}) - \psi(\bar{x}) \right] \right\}^\dagger$$

$$= \left[\psi^\dagger(\bar{x}+a\hat{\mu}) - \psi^\dagger(\bar{x}) \right] \gamma_\mu \gamma_0 \psi(\bar{x}) = - \left[\bar{\psi}(\bar{x}+a\hat{\mu}) - \bar{\psi}(\bar{x}) \right] \gamma_\mu \psi(\bar{x})$$

Summing over $\bar{x}$ and shifting, we equivalently get $\bar{\psi}(\bar{x}) \gamma_\mu \left[\bar{\psi}(\bar{x}) - \bar{\psi}(\bar{x}-a\hat{\mu}) \right]$.

[For scalar theories, $S^{(\text{scalar})} = \sum_{\bar{x}} a^4 \left\{ \frac{1}{a^2} \left[U_\mu(\bar{x}) \bar{\phi}(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x}) \right]^\dagger \left[U_\mu(\bar{x}) \bar{\phi}(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x}) \right] + \ldots \right\}$,
there is no such problem!]

Attempt 2:

The problem can easily be solved by symmetrising the derivative:

$$S = \sum_{\bar{x}} a^4 \, \bar{\psi}(\bar{x}) \left\{ \frac{1}{2a} \gamma_\mu \left[\psi(\bar{x}+a\hat{\mu}) - \psi(\bar{x}-a\hat{\mu}) \right] + m \psi(\bar{x}) \right\}$$

Now it is "Hermitean".

[Or, rather, "anti-Hermitean", if a sign (-1) is inserted upon commutation of the Grassmann variables $\psi, \psi^\dagger$. To be even more precise, the proper concept is "reflection positivity".]

Action in momentum space

As usual, we write $\Psi(\bar{x}) = \int_p e^{ip\cdot x}\,\psi(p)$, $\int_p = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4}$, or with finite V.

$\bar{\Psi}(\bar{x}) = \int_q e^{-iq\cdot x}\,\bar{\Psi}(q)$

[just a convention, due to the complex conjugation in $\Psi^+\gamma_0$]

$$S = \int_{p,q} \underbrace{\sum_{\bar{x}} a^4\, e^{-iq\cdot x + ip\cdot x}}_{\delta(-q+p)}\; \bar{\Psi}(q) \left\{\frac{1}{2a}\gamma_\mu \left(e^{iap_\mu} - e^{-iap_\mu}\right) + m\right\} \Psi(p)$$

$$= \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\; \bar{\Psi}(p) \left\{ i\gamma_\mu \mathring{p}_\mu + m \right\} \Psi(p) \, ;$$

$$\mathring{p}_\mu \equiv \frac{1}{a}\sin ap_\mu \qquad \left(\text{recall: } \tilde{p}_\mu = \frac{2}{a}\sin\frac{ap_\mu}{2}\right)$$

$$= \frac{2}{a}\sin\frac{ap_\mu}{2}\cos\frac{ap_\mu}{2} = \tilde{p}_\mu \sqrt{1 - \frac{a^2}{4}\tilde{p}_\mu^{\,2}} \, .$$

Let us write $D(p) = i\gamma_\mu \mathring{p}_\mu + m\,\mathbb{1}_{4\times4} = 4\times4-\text{matrix}$

Claim: $D^{-1}(p) = \left[-i\gamma_\mu\mathring{p}_\mu + m\,\mathbb{1}_{4\times4}\right]\cdot \frac{1}{\sum_\mu \mathring{p}_\mu^2 + m^2} \equiv \left[-i\mathring{\not{p}} + m\right]\cdot\frac{1}{\mathring{p}^2 + m^2}$

Proof: $D^{-1}(p)\cdot D(p) = \frac{1}{\mathring{p}^2 + m^2}\left\{ -i\mathring{\not{p}}\, i\mathring{\not{p}} \underbrace{- i\mathring{\not{p}} m}_{\sim} \underbrace{+ m i\mathring{\not{p}}}_{\sim} + m^2 \right\}$

$$\mathring{\not{p}}\mathring{\not{p}} = \mathring{p}_\mu\mathring{p}_\nu\gamma_\mu\gamma_\nu = \frac{1}{2}\mathring{p}_\mu\mathring{p}_\nu\{\gamma_\mu,\gamma_\nu\} = \mathring{p}_\mu\mathring{p}_\mu \qquad \square$$

Propagator

What is $\langle \psi_\alpha(p)\,\bar\psi_\beta(q)\rangle_0$, $\alpha\beta = 1,\dots,4$?

In order to find out, we have to recall something about Grassmann integration.

In order to obtain Fermi-Dirac statistics, fermion are treated as Grassmann variables.

Rules for Grassmann integration are very simple:

* $\psi, \bar\psi$ are treated as independent variables (like ϕ, ϕ^* for $\mathrm{Re}\phi$, $\mathrm{Im}\phi$)
* $\int d\psi = \int d\bar\psi = 0$
* $\int d\psi\,\psi = \int d\bar\psi\,\bar\psi = 1$
* $\psi^2 = \bar\psi^2 = 0$; $\psi\bar\psi = -\bar\psi\psi$, $\{\psi, d\psi\} = \{\psi, d\bar\psi\} = \{\bar\psi, d\psi\} = \{\bar\psi, d\bar\psi\} = 0$
* All this applies separately for every $\psi_\alpha(\bar x)$, $\alpha = 1,\dots,4$, $\bar x \in$ lattice; or $\psi_\alpha(\bar p)$.
* Integration measure $\equiv \int \{\prod_{\bar x,\alpha} d\psi_\alpha(\bar x)\}\{\prod_{\bar y,\beta} d\bar\psi_\beta(\bar y)\}$

To find out the general answer, let us look at the example of a 2x2-matrix:

$$M = \begin{pmatrix} a & d \\ c & b \end{pmatrix} \qquad S = \bar\psi M \psi = a\bar\psi_1\psi_1 + b\bar\psi_2\psi_2 + c\,\bar\psi_2\psi_1 + d\,\bar\psi_1\psi_2$$

$$\frac{\langle \psi_1\bar\psi_2\rangle_0}{\langle 1\rangle_0} = \frac{\int d\psi_1 d\psi_2 d\bar\psi_1 d\bar\psi_2 \;\; \psi_1\bar\psi_2 \;\; \exp\left(-a\bar\psi_1\psi_1 - b\bar\psi_2\psi_2 - c\bar\psi_2\psi_1 - d\bar\psi_1\psi_2\right)}{\int d\psi_1 d\psi_2 d\bar\psi_1 d\bar\psi_2 \;\; \exp\left(-a\bar\psi_1\psi_1 - b\bar\psi_2\psi_2 - c\bar\psi_2\psi_1 - d\bar\psi_1\psi_2\right)}$$

The only contribution come from the terms $\sim \psi_1\psi_2\bar\psi_1\bar\psi_2$ in the Taylor expansions!

$$= \frac{\int d\psi_1 d\psi_2 d\bar\psi_1 d\bar\psi_2 \;\; \psi_1\bar\psi_2 \;\; (-d\bar\psi_1\psi_2)}{\int d\psi_1 d\psi_2 d\bar\psi_1 d\bar\psi_2 \left(+ab\,\bar\psi_1\psi_1\bar\psi_2\psi_2 + cd\,\bar\psi_2\psi_1\bar\psi_1\psi_2\right)}$$

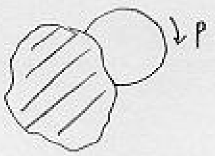
$$= \frac{-d\cdot(-1)^? \int d\psi_1\psi_1 \int d\psi_2\psi_2 \int d\bar\psi_1\bar\psi_1 \int d\bar\psi_2\bar\psi_2}{\left[+ab(-1)^1 \;\; +cd(-1)^0 \right] \int d\psi_1\psi_1 \int d\psi_2\psi_2 \int d\bar\psi_1\bar\psi_1 \int d\bar\psi_2\bar\psi_2}$$

$$= \frac{-d}{ab - cd} \qquad ; \qquad M^{-1} = \frac{1}{ab - cd}\begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$$

$$\Rightarrow \frac{\langle \psi_1\bar\psi_2\rangle_0}{\langle 1\rangle_0} = \left[M^{-1}\right]_{12} \quad \Rightarrow \quad \langle \psi_\alpha(p)\bar\psi_\beta(q)\rangle_0 = \delta(p-q)\cdot\frac{\left[-i\not{p} + m\right]_{\alpha\beta}}{p^2 + m^2}$$

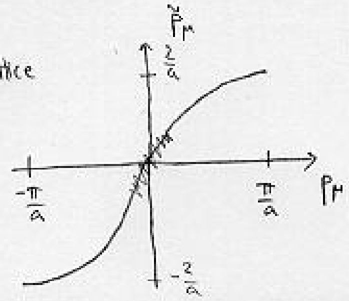
Back to the continuum limit

Let us now think in terms of the weak coupling expansion, and consider some loop with the fermion propagator



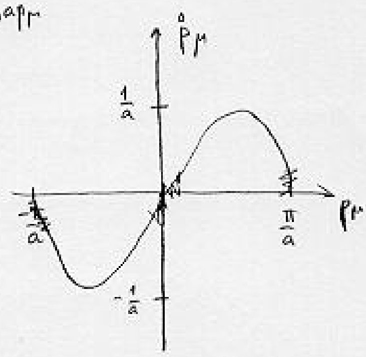
* Integration range: $p_\mu \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right)$ (Brillouin zone)

* Bosonic (scalar, gauge field) lattice momenta: $\tilde{p}_\mu = \frac{2}{a}\sin\frac{a p_\mu}{2}$



A continuum-like contribution comes from $p_\mu \approx 0$, $\tilde{p}_\mu \approx p_\mu$.

* Fermionic lattice momenta: $\mathring{p}_\mu = \frac{1}{a}\sin a p_\mu$



$\Rightarrow$ there are two regions where the behaviour is continuum-like !! This is called the fermion doubling problem. The same is true for each axis, and thus in four dimensions a naively discretised fermion in fact represents $2^4 = 16$ continuum fermions, rather than one as we wanted.

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