Gauge + Higgs theory

So far we have considered two types of models:

* "scalar" models, with degrees of freedom $\bar{\phi}$ living on the sites:

$$S^{(\text{scalar})} = \sum_{\bar{x}} a^4 \left[\frac{1}{2} \sum_{\mu=0}^{3} \frac{|\bar{\phi}(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x})|^2}{a^2} + \frac{1}{2} m_0^2 |\bar{\phi}(\bar{x})|^2 + \frac{1}{4} \lambda_0 |\bar{\phi}(\bar{x})|^4 \right], \quad \bar{\phi} \in \mathbb{R}^N .$$

* pure gauge theory, with degrees of freedom U_μ living on the links:

$$S^{(\text{gauge})} = \frac{1}{g_0^2} \sum_{\bar{x}} \sum_{\mu\nu=0}^{3} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(\bar{x}) \right], \qquad P_{\mu\nu} \in SU(N_c) .$$

Let us now construct a model with both types of degrees of freedom, respecting the principle of gauge invariance!

Gauge transformation property of links: $\quad U_\mu(\bar{x}) \to U'_\mu(\bar{x}) = g(\bar{x}) U_\mu(\bar{x}) g^{-1}(\bar{x}+a\hat{\mu}), \quad g(\bar{x}) \in SU(N_c).$

$$\bar{x} \xrightarrow{\;U_\mu\;} \bar{x}+a\hat{\mu}$$

We generalise $\bar{\phi}$ to be complex-valued N_c-component vectors, $\bar{\phi} \in \mathbb{C}^{N_c}$, and require simply a local transformation law,

$$\bar{\phi}(\bar{x}) \to \bar{\phi}'(\bar{x}) = g(\bar{x})\, \bar{\phi}(\bar{x}) .$$

This is just a generalization of the invariance of QED in phase transformation:

$$\phi(\bar{x}) \to \phi'(\bar{x}) = e^{i\alpha(\bar{x})} \phi(\bar{x}), \qquad e^{i\alpha(\bar{x})} \in U(1) .$$

It is said that scalar fields with these properties transform under the "<u>fundamental representation</u>" of the gauge group.

Scalar fields are also often referred to as <u>Higgs fields</u>.

["Adjoint representation": $\bar{\phi} \in \mathbb{R}^{N_c^2-1}$; write the components ϕ^a into a traceless Hermitean matrix $\Phi = \phi^a T^a$. The transformation law is then $\Phi(\bar{x}) \to \Phi'(\bar{x}) = g(\bar{x}) \Phi(\bar{x}) g^{-1}(\bar{x})$.]

Invariant operators

- The part $S^{(\text{gauge})}$ is already gauge invariant.
- The terms $|\bar\phi|^2$, $|\bar\phi|^4$ in $S^{(\text{scalar})}$ are easily made so:

conventional !

$$|\bar\phi|^2 \Rightarrow 2\bar\phi^+\bar\phi$$

Then $\bar\phi^+\bar\phi \to \bar\phi'^+\bar\phi' = \bar\phi^+ g^+(x)g(x)\bar\phi = \bar\phi^+\bar\phi$.

- How about the cross terms?

$$\sum_{\bar x}\sum_{\mu=0}^{3} |\bar\phi(\bar x+a\hat\mu) - \bar\phi(\bar x)|^2 = \sum_{\bar x}\left\{ 8|\bar\phi(\bar x)|^2 - 2\sum_{\mu=0}^{3}\bar\phi(\bar x)\cdot\bar\phi(\bar x+a\hat\mu)\right\}$$

$$\Rightarrow \sum_{x}\left\{ 8\bar\phi^+(\bar x)\bar\phi(\bar x) - 2\sum_{\mu=0}^{3}\bar\phi^+(\bar x)\,U_\mu(\bar x)\,\bar\phi(\bar x+a\hat\mu)\right\}$$

Of course one could equally well write the cross term as

$$\bar\phi(\bar x+a\hat\mu)\cdot\bar\phi(\bar x) \Rightarrow \bar\phi^+(\bar x+a\hat\mu)\,U^+_\mu(\bar x)\,\phi(\bar x)$$

To keep the action real, it's best to take the average of the two:

$$\Rightarrow \sum_{\bar x}\sum_{\mu=0}^{3}\left\{ 2\bar\phi^+(\bar x)\bar\phi(\bar x) - \bar\phi^+(\bar x)U_\mu(\bar x)\bar\phi(\bar x+a\hat\mu) - \bar\phi^+(\bar x+a\hat\mu)U^+_\mu(\bar x)\phi(\bar x)\right\}$$

$$= \sum_{\bar x}\sum_{\mu=0}^{3}\left\{ 2\bar\phi^+(\bar x)\bar\phi(\bar x) - 2\,\mathrm{Re}\left[\bar\phi^+(\bar x)U_\mu(\bar x)\bar\phi(\bar x+a\hat\mu)\right]\right\}$$

$$= \sum_{\bar x}\sum_{\mu=0}^{3}\left[U_\mu(\bar x)\bar\phi(\bar x+a\hat\mu) - \bar\phi(\bar x)\right]^+\left[U_\mu(\bar x+a\hat\mu)\bar\phi(\bar x+a\hat\mu) - \bar\phi(\bar x)\right]$$

$$\Rightarrow\quad S^{(\text{scalar})} = \sum_{\bar x} a^4\left\{\frac{2}{a^2}\sum_{\mu=0}^{3}\left[\bar\phi^+(\bar x)\bar\phi(\bar x) - \mathrm{Re}\,\bar\phi^+(\bar x)U_\mu(\bar x)\bar\phi(\bar x+a\hat\mu)\right] + m_0^2\,\bar\phi^+(\bar x)\bar\phi(\bar x) + \lambda_0\left(\bar\phi^+(\bar x)\phi(\bar x)\right)^2\right\}$$

Approach to the continuum limit

We expand again $U_\mu(\bar{x}) = e^{iag_0 A_\mu(\bar{x})} \approx 1 + iag_0 A_\mu(\bar{x}) + \mathcal{O}((ag_0)^2)$.

$$\Rightarrow \frac{1}{a}\left[U_\mu(\bar{x})\,\bar{\phi}(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x})\right]$$

$$\approx \frac{1}{a}\left[\bar{\phi}(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x})\right] + ig_0 A_\mu(\bar{x})\left[\bar{\phi}(\bar{x}) + a\hat{\Delta}_\mu\bar{\phi}(\bar{x})\right] + \mathcal{O}(a^2)$$

$$\approx \underbrace{\left[\partial_\mu + ig_0 A_\mu(\bar{x})\right]}\bar{\phi}(\bar{x}) + \mathcal{O}(a)$$

This is the covariant derivative, $D_\mu = \partial_\mu + ig_0 A_\mu(\bar{x})$, of continuum field theory.

Thus, in the ("naive") continuum limit,

$$S^{(\text{gauge})} + S^{(\text{scalar})} \longrightarrow \int d^4x \left\{ \frac{1}{4} F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\bar{\phi})^\dagger(D_\mu\bar{\phi}) + m_0^2\bar{\phi}^\dagger\bar{\phi} + \lambda_0(\bar{\phi}^\dagger\bar{\phi})^2 \right\}.$$

This is an essential part of the action describing weak interactions, if $N_c = 2$.

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Like for pure gauge theory, a proper way to approach the continuum limit is not the "naive" expansion in a, but a weak coupling expansion in the parameter g_0. At order $\mathcal{O}(g_0^0)$, this leads to

$$S^{(\text{scalar})} = \sum_x a^4 \left\{ \frac{1}{a^2}\sum_{\mu=0}^{4} \left[\bar{\phi}^\dagger(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x})\right]^\dagger\left[\bar{\phi}(\bar{x}+a\hat{\mu}) - \bar{\phi}(\bar{x})\right] + m_0^2\bar{\phi}^\dagger\bar{\phi} + \lambda_0(\bar{\phi}^\dagger\bar{\phi})^2 \right\}$$

By writing $\bar{\phi} = \frac{1}{\sqrt{2}}(\bar{\phi}_1 + i\bar{\phi}_2)$, this is precisely of the form on p. 19-20, and leads to the usual lattice propagators, such as

$$\frac{\langle \phi_1^a(p_1)\,\phi_1^b(p_2)\rangle_0}{\langle 1\rangle_0} = \delta^{ab}\cdot V\cdot \delta_{p_1+p_2,0}\,\frac{1}{\sum_{\mu=0}^{3}\tilde{p}_\mu^2 + m_0^2}.$$

"Gauge symmetry breaking"

The parameter λ_0 must be non-negative, $\lambda_0 \geq 0$; otherwise the integral over $\bar{\phi}$ is not convergent. The mass parameter m_0^2 could in principle have any sign, however.

Without the link variables U_μ (or more precisely, for $U_\mu = \mathbb{1}$), changing m_0^2 from very large values $\gg 0$ to very small values $\ll 0$ leads to a symmetry breaking phase transition at some $m_0^2 = m_{0,c}^2$. We already discussed this for spin models in the beginning. (p. 6-10)

Let us look at the action S for constant field configurations

= "zero-momentum modes"

= "mean fields". (mean = average)

Formally:
$$Z = \int \Big\{ \prod_{\bar{x},\mu} \mathcal{D}\bar{\phi}(\bar{x})\, \mathcal{D}U_\mu(\bar{x}) \Big\} e^{-S} = \int_{-\infty}^{\infty} d\bar{v} \; \underbrace{\int \Big\{ \prod_{\bar{x},\mu} \mathcal{D}\bar{\phi}(\bar{x})\, \mathcal{D}U_\mu(\bar{x}) \Big\}\, \delta\Big(\bar{v} - \frac{1}{V}\sum_{\bar{x}} a^4 \bar{\phi}(\bar{x})\Big) e^{-S}}_{\equiv\, e^{-\Gamma(\bar{v})}} \;, \quad \bar{v} \in \mathbb{C}^{N_c}$$

If U_μ's are constant as well, $P_{\mu\nu} = \mathbb{1}$, and then at leading order (no integration)

$$\Gamma(\bar{v}) = S(\bar{v}) = V \cdot \Big\{ \frac{2}{a^2} \sum_\mu \big[\bar{v}^\dagger \bar{v} - \mathrm{Re}\, \bar{v}^\dagger U_\mu \bar{v} \big] + m_0^2 \bar{v}^\dagger \bar{v} + \lambda_0 (\bar{v}^\dagger \bar{v})^2 \Big\}$$

volume, $V = \sum_{\bar{x}} a^4$

$$= V \cdot \Big\{ \frac{2}{a^2} \sum_\mu \bar{v}^\dagger \Big[\mathbb{1} - \frac{U_\mu + U_\mu^\dagger}{2} \Big] \bar{v} + m_0^2 \bar{v}^\dagger \bar{v} + \lambda_0 (\bar{v}^\dagger \bar{v})^2 \Big\}$$

Action is minimised (= has saddle point) at $U_\mu = \mathbb{1}$: Consider for simplicity $N_c = 2$.

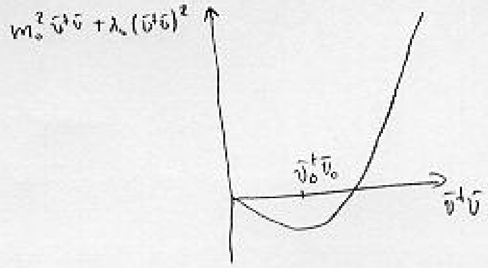
$$U_\mu = e^{i\phi^a T^a} \rightarrow \frac{U_\mu + U_\mu^\dagger}{2} = \mathbb{1} - \frac{1}{4}\phi^a\phi^b \{T^a, T^b\} \quad ; \quad \mathbb{1} - \frac{U_\mu + U_\mu^\dagger}{2} = + \frac{1}{4}\phi^a\phi^b \{T^a, T^b\}$$

Pauli matrix

For $SU(2)$, $T^a = \frac{\sigma^a}{2}$, $\{T^a, T^b\} = \frac{1}{2}\delta^{ab}$

How to minimise $m_0^2 \bar{v}^+\bar{v} + \lambda_0 (\bar{v}^+\bar{v})^2$, if $m_0^2 < 0$?

[In reality the point is not at $m_0^2 = 0$, because the integral in the definition of $\Gamma(\bar{v})$ gives corrections to the approximation $\Gamma(\bar{v}) \approx S(\bar{v})$, but the arguments remain the same.]



$\Rightarrow$ We may choose some representative for $\bar{v}_0$, such that S is minimised. This appears, however, to "break gauge invariance", because the choice is not invariant in $\bar{v}_0 \to g(\bar{x})\,\bar{v}_0$. But obviously the invariance is still there in the action and in physics, where only $\bar{v}^+\bar{v}$ and other gauge-invariant objects appear: the breaking of gauge invariance is just some (historically) (badly chosen) jargon.

The physical consequence of "gauge symmetry breaking", $\bar{v}_0^+\bar{v}_0 > 0$ $(N_c = 2)$:

$$U_\mu = e^{ia g_0 A_\mu^a T^a} \quad \Rightarrow \quad \frac{2}{a^2} \sum_\mu \bar{v}_0^+ \left[\mathbb{1} - \frac{U_\mu + U_\mu^+}{2} \right] \bar{v}_0$$

$$\approx \frac{1}{4} g_0^2\, \bar{v}_0^+\bar{v}_0 \cdot \sum_\mu A_\mu^a A_\mu^a$$

Like $\frac{1}{2} m_0^2 \phi_0^2$ is a mass term for a real scalar field ϕ_0, this term is interpreted (in the continuum limit $g_0 \to 0$) as a mass squared $m_W^2 = \frac{1}{2} g_0^2\, \bar{v}_0^+\bar{v}_0$ for the gauge bosons, $W^\pm, Z^0$.

This mechanism for mass generation is called the "Higgs mechanism".
