Strong coupling expansion

$$S = \frac{1}{g_0^2} \sum_{\bar{x}} \sum_{\mu,\nu} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(\bar{x}) \right] ,$$

$$P_{\mu\nu}(\bar{x}) = U_\mu(\bar{x})\, U_\nu(\bar{x}+a\hat{\mu})\, U_\mu^\dagger(\bar{x}+a\hat{\nu})\, U_\nu^\dagger(\bar{x}) \quad , \qquad U_\mu(\bar{x}) \in SU(N_c)$$

Weak coupling expansion $\hat{=}$ $g_0^2 \to 0$ (continuum limit).

Strong —"— $\hat{=}$ $g_0^2 \to \infty$.

Recalling $P_{\nu\mu} = P_{\mu\nu}^{-1} = P_{\mu\nu}^\dagger$ and noticing $P_{\mu\mu}(\bar{x}) = \mathbb{1}$, the action can be written as

$$\exp(-S) = \text{const} \times \exp\left[\frac{1}{g_0^2} \sum_{\bar{x}} \sum_{\mu<\nu} \mathrm{Tr}\left(P_{\mu\nu}(\bar{x}) + P_{\mu\nu}^\dagger(\bar{x}) \right) \right] .$$

This is now expanded in a power series in $\frac{1}{g_0^2}$. We will need to consider integrals of the type

$$f(p,q) = \int dU_\mu(\bar{x})\, [U_\mu(\bar{x})]^p\, [U_\nu^\dagger(\bar{x})]^q \quad , \quad f(0,0) = 1 .$$

$f(1,0) = f(0,1) = 0$! [recall that $\langle \Theta \rangle = 0$ if $\Theta \to g\Theta$; p. 70]

$f(1,1) = 0$, if $\nu \neq \mu$

For $f(1,1)$, have to look at the components:

$$I_{ijkl} = \int dU_\mu(\bar{x})\, [U_\mu(\bar{x})]_{ij}\, [U_\mu^\dagger(\bar{x})]_{kl}$$

gauge transformation

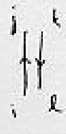
$$\overset{\downarrow}{=} g(\bar{x})_{i\alpha}\, g^{-1}(\bar{x}+a\hat{\mu})_{\beta j}\, g(\bar{x}+a\hat{\mu})_{k\gamma}\, g^{-1}(\bar{x})_{\delta l} \cdot I_{\alpha\beta\gamma\delta}$$

For this to be invariant, have to require $I_{\alpha\beta\gamma\delta} \propto \delta_{\alpha\delta}\, \delta_{\beta\gamma}$!

To fix the proportionality constant, write:

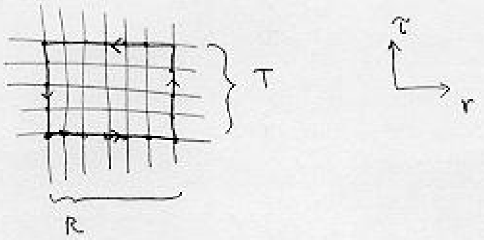
$$c \cdot \delta_{il}\, \delta_{jk} = \int dU_\mu(\bar{x})\, [U_\mu(\bar{x})]_{ij}\, [U_\mu^\dagger(\bar{x})]_{kl} \qquad \Big|\cdot \delta_{jk}$$

$$c \cdot \delta_{il} \cdot N_c = \int dU_\mu(\bar{x})\, \delta_{il} = \delta_{il} \qquad \Rightarrow \qquad \underline{\underline{c = \frac{1}{N_c}}}$$

$$\int dU_\mu$$  $$= \frac{1}{N_c}$$ 

Observables:

Wilson loop: Rather than a single plaquette, consider now a big rectangular loop:



$$W(R,T) \equiv \mathrm{Tr}\left\{ \prod_{\substack{\text{links around}\\ \text{the loop}}} U_\mu \right\}$$

Static potential:

$$V(R) \equiv -\lim_{T\to\infty} \frac{1}{T} \ln \langle W(R,T) \rangle$$

, assuming that lattice is infinite in τ-direction.

Interpretation: A static "quark" and "antiquark" are sitting a distance R apart and "propagate" (or are parallel transported) in time.

The expectation value for this to happen is

$$W(R,T) = C \cdot \exp\left(-T \cdot V(R)\right),$$

where $V(R)$ is related to the energy of such a configuration.

In weak coupling expansion:

$$V(R) = -\frac{N_c^2-1}{2N_c} \cdot \frac{g_0^2}{4\pi R}$$

[exercise!]

group-theoretical factor

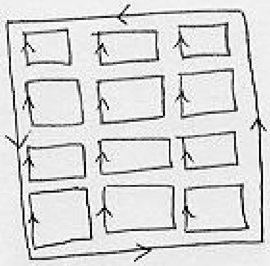
Coulomb-potential!

How can this observable now be computed?

* $$Z = \int \left\{ \prod_{x,\mu} dU_\mu(x) \right\} \exp(-S) = \text{const} \times \int \left\{ \prod_{x,\mu} dU_\mu(x) \right\} \left\{ 1 + O\left(\frac{1}{g_0^2}\right) \right\} = \text{const.}$$

* For the Wilson loop we have learned that only integrals of the type $\int dU_\mu(\vec{x}) \, [U_\mu(\vec{x})][U_\mu^\dagger(\vec{x})]$ are non-zero. Therefore have to cover, or "tile", the inside of the loop with elementary plaquettes!

Need $\frac{R \cdot T}{a^2}$ plaquettes!



$$\Rightarrow \langle W(R,T) \rangle = c \cdot \left(\frac{1}{g_0^2} \right)^{\frac{RT}{a^2}} \quad !$$

* How to find out constant c?

  - p.76 $\Rightarrow$ each link gives a factor $\frac{1}{N_c}$; there are $\frac{RT}{a^2}$ plaquettes, each has two independent links, and additionally there are $\frac{R}{a} + \frac{T}{a}$ from the "opposite boundary":
  $$\left(\frac{1}{N_c} \right)^{2\frac{RT}{a^2} + \frac{R}{a} + \frac{T}{a}}$$

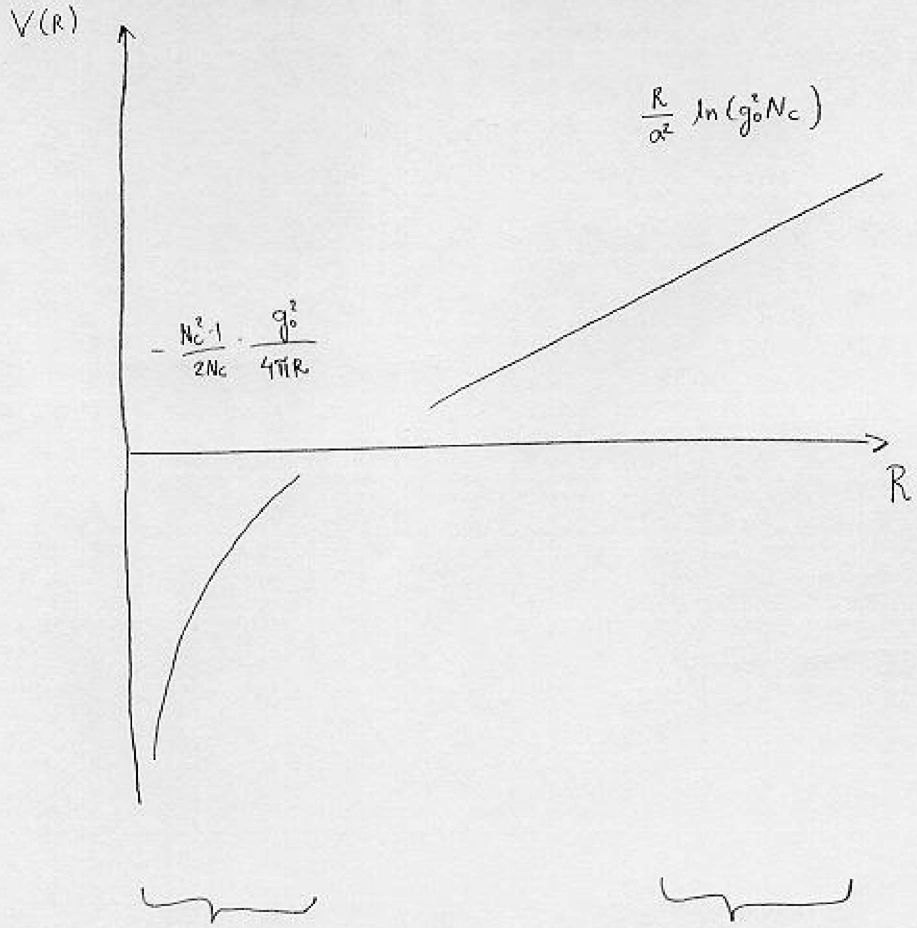
  - Link integrations act as: $[U^\dagger]_{kl} [U]_{ij} \rightarrow \frac{1}{N_c} \delta_{kj} \delta_{li}$

    Thus, there is finally a sum left over matrix indices on sites:
    $$\left(\frac{1}{N_c} \right)^{2\frac{RT}{a^2} + \frac{R}{a} + \frac{T}{a}} \times \begin{Bmatrix} \circlearrowleft & \circlearrowleft & \circlearrowleft & \circlearrowleft \\ \circlearrowleft & \circlearrowleft & \circlearrowleft & \circlearrowleft \\ \circlearrowleft & \circlearrowleft & \circlearrowleft & \circlearrowleft \\ \circlearrowleft & \circlearrowleft & \circlearrowleft & \circlearrowleft \\ \circlearrowleft & \circlearrowleft & \circlearrowleft & \circlearrowleft \end{Bmatrix} = \left(\frac{1}{N_c} \right)^{2\frac{RT}{a^2} + \frac{R}{a} + \frac{T}{a}} \cdot (N_c)^{\left(\frac{R}{a}+1\right)\left(\frac{T}{a}+1\right)}$$
    $$= \left(\frac{1}{N_c} \right)^{\frac{RT}{a^2} - 1}$$

$$\Rightarrow \langle W(R,T) \rangle = N_c \cdot \left(\frac{1}{g_0^2 N_c} \right)^{\frac{RT}{a^2}}$$

$$\Rightarrow V(R) = - \lim_{T \to \infty} \frac{1}{T} \ln \langle W(R,T) \rangle = \frac{R}{a^2} \ln \left(g_0^2 N_c \right)$$

We can now plot the potential:



Small R:
result indep. of a
⇒ approach cont. limit
⇒ g_0^2 is small
⇒ weak coupling justified

large R:
keep a non-zero
⇒ g_0^2 large
⇒ strong coupling justified

Because the potential rises without a limit, the system will <u>only possess bound states</u>! [No continuum states like for the Coulomb potential alone.]

This is called <u>confinement</u>, and should explains why no free quarks are observed in nature.

Dilemma:

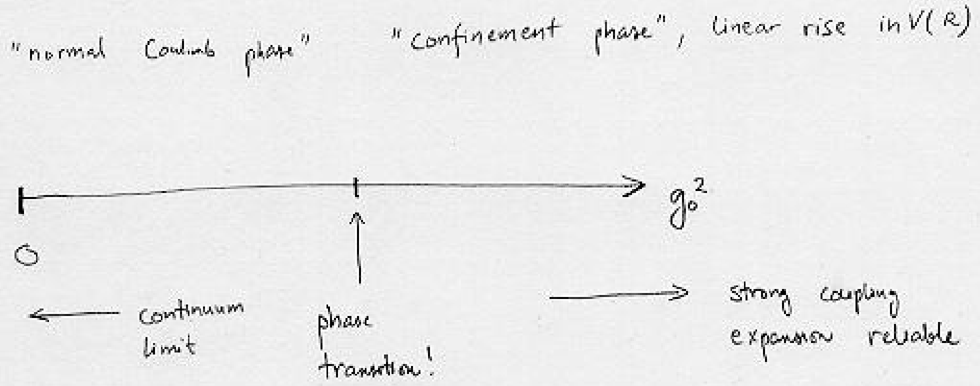
The confining potential was obtained in the strong coupling expansion. But close to the continuum limit $g_0^2 \to 0$!

Does the linearly rising part of the potential remain there even if the method with which we obtained it, breaks down?

Numerically, with lattice Monte Carlo: YES!

Analytically: has not been proven. This is one of the "Millennium problems" of mathematics [http://www.claymath.org/Millennium_Prize_Problems/]

A counterexample: "U(1) gauge theory" [$U_\mu(x) \in U(1)$ rather than $SU(N_c)$]



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