Weak coupling expansion

$$S = \frac{\beta_G}{2N_c} \sum_{\bar{x}} \sum_{\mu,\nu=0}^{3} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(\bar{x}) \right] ,$$

$$P_{\mu\nu}(\bar{x}) = U_\mu(\bar{x}) U_\nu(\bar{x}+a\hat{\mu}) U_\mu^{-1}(\bar{x}+a\hat{\nu}) U_\nu^{-1}(\bar{x}) \quad , \quad U_\mu(\bar{x}) \in SU(N_c) , \; U_\mu^{-1} = U_\mu^{+} .$$

The theory depends on a single parameter, β_G.
"Weak coupling expansion" ≡ limit of <u>large β_G</u>.

As we have seen, $SU(N_c)$ matrices can be parametersised in term of the generators T^a, $a=1,\dots,N_c^2-1$, as

$$U_\mu(\bar{x}) = e^{i\phi_\mu(\bar{x})} \quad , \quad \phi_\mu(\bar{x}) = \phi_\mu^a(\bar{x}) T^a$$

(Einstein summation convention will be assumed in the following.)

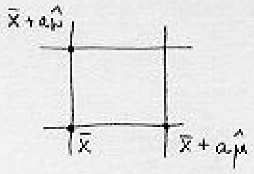
$$P_{\mu\nu}(\bar{x}) = e^{i\phi_{\mu\nu}(\bar{x})} \quad , \quad \phi_{\mu\nu}(\bar{x}) = \phi_{\mu\nu}^a(\bar{x}) T^a$$

$$\phi_\mu^a , \phi_{\mu\nu}^a \in \mathbb{R} .$$

This means that $\mathrm{Tr}\, P_{\mu\nu} \leq \mathrm{Tr}\, \mathbb{1} \Rightarrow \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}\right] \geq 0$.
Action (~ energy) wants to be minimised
⇒ for $\beta_G \to \infty$, $\mathrm{Tr}\, P_{\mu\nu}$ wants to be as close to $\mathrm{Tr}\, \mathbb{1}$ as possible.

⇒ weak coupling expansion: <u>expand U_μ, $P_{\mu\nu}$ in terms of ϕ_μ, $\phi_{\mu\nu}$</u> !

The weak coupling expansion also tells how to approach the continuum limit !



As $a \to 0$, the points $\bar{x}$, $\bar{x}+a\hat{\mu}$ are almost the same, and "parallel transport" of information should become trivial.

Let us therefore write:

$$\phi^a_\mu(\bar{x}) \equiv \underbrace{a}_{\text{lattice spacing}} \cdot \underbrace{g_0}_{\text{bare gauge coupling}} \cdot \underbrace{A^a_\mu(\bar{x})}_{\text{"Continuum" gauge field}}$$

$$\phi^a_\nu(\bar{x}+a\hat{\mu}) = a g_0 A^a_\nu(\bar{x}+a\hat{\mu}) \qquad \text{etc}$$

Let us work out $\phi_{\mu\nu}$ to second order in a !

Need to do some matrix algebra:

$$e^{B_1} e^{B_2} e^{B_3} e^{B_4} = e^{M(B)}$$

Using the Campbell-Baker-Hausdorff formula,

$$M(B) = \sum_i B_i + \frac{1}{2} \sum_{i<j} [B_i, B_j] + \mathcal{O}(B^3)$$

Check by an explicit series expansion for the case $e^{B_1} e^{B_2}$:

LHS: $\left(1 + B_1 + \frac{1}{2} B_1^2\right)\left(1 + B_2 + \frac{1}{2} B_2^2\right) = 1 + B_1 + B_2 + \frac{1}{2} B_1^2 + B_1 B_2 + \frac{1}{2} B_2^2$

RHS: $1 + (B_1 + B_2) + \frac{1}{2}[B_1, B_2] + \frac{1}{2}(B_1 + B_2)^2$

$= 1 + B_1 + B_2 + \frac{1}{2} B_1^2 + \frac{1}{2} B_2^2 + \frac{1}{2}\left(B_1 B_2 - \cancel{B_2 B_1} + B_1 B_2 + \cancel{B_2 B_1}\right)$ OK!

Therefore we obtain :

$$\phi_{\mu\nu}(\bar{x}) = a g_0 \Big[\underset{\substack{\parallel \\ \Theta_1}}{A_\mu(\bar{x})} + \underset{\substack{\parallel \\ \Theta_2}}{A_\nu(\bar{x}+a\bar{\mu})} - \underset{\substack{\parallel \\ \Theta_3}}{A_\mu(\bar{x}+a\bar{\nu})} - \underset{\substack{\parallel \\ \Theta_4}}{A_\nu(\bar{x})} \Big] + \frac{i}{2}(a g_0)^2 \sum_{i<j} \left[\Theta_i, \Theta_j\right]$$

Expanding formally $A_\nu(\bar{x}+a\bar{\mu}) \equiv A_\nu(\bar{x}) + a\,\hat{\Delta}_\mu A_\nu(\bar{x})$, where $\hat{\Delta}_\mu A_\nu(\bar{x}) = \frac{1}{a}\left[A_\nu(\bar{x}+a\bar{\mu}) - A_\nu(\bar{x})\right]$, we get

$$\phi_{\mu\nu}(\bar{x}) = a^2 g_0\, F_{\mu\nu}(\bar{x}) + \mathcal{O}(a^3) ,$$

where

$$F_{\mu\nu}(\bar{x}) = \hat{\Delta}_\mu A_\nu(\bar{x}) - \hat{\Delta}_\nu A_\mu(\bar{x}) + \frac{i}{2} g_0 \Big\{ [\cancel{A_\mu}, A_\nu] + [\cancel{A_\nu}, -A_\mu] - [\cancel{A_\mu, A_\nu}]$$
$$+ [A_\nu, -A_\mu] + [\cancel{A_\mu, -A_\nu}]$$
$$+ [A_\mu, A_\nu] \Big\}$$

$$= \hat{\Delta}_\mu A_\nu(\bar{x}) - \hat{\Delta}_\nu A_\mu(\bar{x}) + i g_0 \left[A_\mu, A_\nu\right]$$

In component form $(F_{\mu\nu} = F^a_{\mu\nu} T^a)$ and in continuum limit $(\hat{\Delta}_\mu \to \partial_\mu)$ we get

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_0 f^{abc} A^b_\mu A^c_\nu \quad ; \qquad F^a_{\nu\mu} = - F^a_{\mu\nu} .$$

Then

$$S = \frac{\beta_G}{2N_c} \sum_{\bar{x}} \sum_{\mu\nu} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}\right] = \frac{\beta_G}{2N_c} \sum_{\bar{x}} \sum_{\mu\nu} \frac{1}{2} a^4 g_0^2\, \mathrm{Tr}\left[F^2_{\mu\nu}\right]$$

$$= \frac{\beta_G g_0^2}{2N_c} \cdot \sum_{\bar{x}} a^4 \sum_{\mu\nu} \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$$

If we choose $\beta_G \equiv \frac{2N_c}{g_0^2}$, this is the continuum action of pure $SU(N_c)$ gauge theory, first written down by C.N. Yang & R.L. Mills, Phys. Rev. 96 (1954) 191!

[The action S is worked out to order A^4 by H.J. Rothe, "Lattice gauge theories", Chapter 15.]

Measure term

If we parameterise the action in terms of the fields ϕ_μ, or A^a_μ, the measure term should also be expressed in this basis.
For illustration, let us work out the first non-trivial order in the weak coupling expansion.

P. 69: if $U_\mu = \exp(i\omega^a T^a)$, then $\int dU_\mu = C \cdot \sqrt{\det g} \cdot \int \prod_{a=1}^{N_c^2-1} d\omega^a$,

where $g_{ab} = \mathrm{Tr}\left[\frac{\partial U_\mu}{\partial \omega^a}\frac{\partial U_\mu^+}{\partial \omega^b}\right] = \mathrm{Tr}\left[U_\mu^+ \frac{\partial U_\mu}{\partial \omega^a}\frac{\partial U_\mu^+}{\partial \omega^b} U_\mu\right]$

Since $U_\mu^+ U_\mu = \mathbb{1}$, $\frac{\partial U_\mu^+}{\partial \omega^a} U_\mu = -U_\mu^+ \frac{\partial U_\mu}{\partial \omega^a}$ $\Rightarrow$ $g_{ab} = -\mathrm{Tr}\left[U_\mu^+ \frac{\partial U_\mu}{\partial \omega^a} \cdot U_\mu^+ \frac{\partial U_\mu}{\partial \omega^b}\right]$

$$U_\mu = \mathbb{1} + i\omega - \frac{1}{2}\omega^2 - \frac{i}{6}\omega^3 + \ldots \;;\; U_\mu^+ = \mathbb{1} - i\omega - \frac{1}{2}\omega^2 + \frac{i}{6}\omega^3 + \ldots$$

$$\frac{\partial U_\mu}{\partial \omega^a} = iT^a - \frac{1}{2}(T^a\omega + \omega T^a) - \frac{i}{6}(T^a\omega^2 + \omega T^a \omega + \omega^2 T^a) + \ldots$$

$$U_\mu^+ \frac{\partial U_\mu}{\partial \omega^a} = iT^a - \frac{1}{2}(T^a\omega + \omega T^a) - \frac{i}{6}(T^a\omega^2 + \omega T^a\omega + \omega^2 T^a)$$
$$+ \quad \omega T^a \quad + \frac{i}{2}(\omega T^a \omega + \omega^2 T^a)$$
$$- \frac{i}{2}(\omega^2 T^a)$$

$$= iT^a + \frac{1}{2}(\omega T^a - T^a\omega) - \frac{i}{6}(T^a\omega^2 - 2\omega T^a\omega + \omega^2 T^a)$$

$$g_{ab} = -\mathrm{Tr}\left\{\left(iT^a - \frac{1}{2}[T^a,\omega] - \frac{i}{6}[[T^a,\omega],\omega]\right)\left(iT^b - \frac{1}{2}[T^b,\omega] - \frac{i}{6}[[T^b,\omega],\omega]\right)\right\}$$

$$= \frac{\delta^{ab}}{2} - \frac{1}{6}\mathrm{Tr}\, T^a[[T^b,\omega],\omega] - \frac{1}{4}\mathrm{Tr}[T^a,\omega][T^b,\omega] - \frac{1}{6}\mathrm{Tr}\, T^b[[T^a,\omega],\omega]$$

$$\det(\mathbb{1} + A) \approx 1 + \mathrm{Tr} A$$

$$\Leftrightarrow \sqrt{\det g} \propto 1 - \sum_{a=1}^{N_c^2-1}\left\{\frac{1}{4}\mathrm{Tr}[T^a,\omega][T^a,\omega] + \frac{1}{3}\mathrm{Tr}\, T^a[[T^a,\omega],\omega]\right\}$$

exercise!

$$\overset{\curvearrowright}{\simeq} 1 - \frac{N_c}{12}\mathrm{Tr}[\omega^2]$$

$\omega^a = a g_0 A^a_\mu$

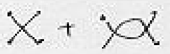
$$\overset{\curvearrowright}{\simeq} \exp\left(-\frac{N_c}{24} \cdot a^2 g_0^2 \, A^a_\mu A^a_\mu\right)$$

$$\delta S = \sum_{\bar{x}} a^4 \cdot \left\{\frac{N_c}{24} \cdot \frac{g_0^2}{a^2} \, A^a_\mu(\bar{x}) A^a_\mu(\bar{x})\right\}$$

"lattice mass counterterm"

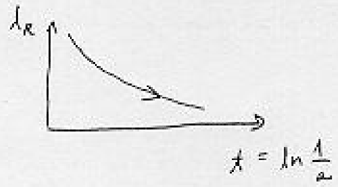
Asymptotic freedom

On p. 60, we studied ϕ^4 - theory in weak coupling expansion.



$$\Rightarrow \quad a \frac{\partial}{\partial a} \lambda_R \Big|_{\lambda_0} = + \frac{9}{8\pi^2} \lambda_R^2$$

$\Rightarrow$



$\Rightarrow$ triviality.

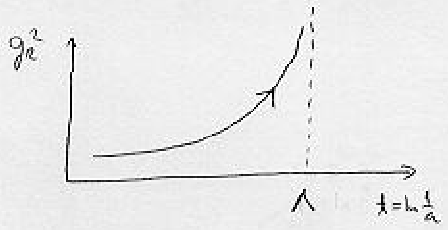
A similar computation can be carried out for pure gauge theory in the weak coupling expansion, after a suitable definition of g_R^2.

Continuum: D. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343
H. Politzer, Phys. Rev. Lett. 30 (1973) 1346

Lattice: A. Hasenfratz, P. Hasenfratz, Phys. Lett. B 93 (1980) 165, and refs. therein.

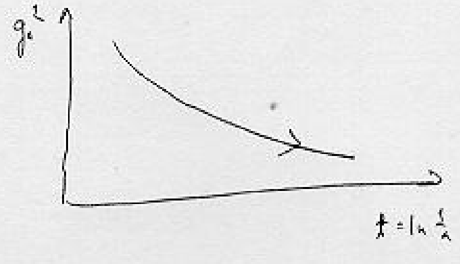
$$\Rightarrow \quad a \frac{\partial}{\partial a} g_R^2 \Big|_{g_0^2} = - \frac{11 N_c}{24\pi^2} g_R^4$$

$\Rightarrow$



$\Rightarrow$ theory is <u>strongly interacting</u>.

Or conversely: $a \frac{\partial}{\partial a} g_0^2 \Big|_{g_R^2} = + \frac{11 N_c}{24\pi^2} g_0^4$



$\Rightarrow$ if lattice spacing is decreased and physics (g_R^2) is kept fixed, the bare coupling g_0^2 gets smaller

$\equiv$

~ asymptotic freedom &

limit exists for $g_0^2 \to 0$!