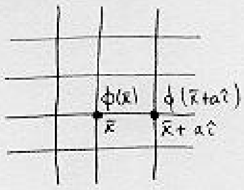
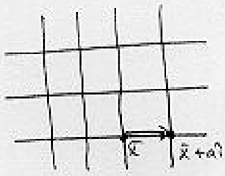
Gauge symmetry & pure gauge theory on the lattice

So far we considered models where the field / spin variables are vectors living on the sites $\bar{x}$: $\longrightarrow$



We will next consider a different type of a model where the variables are <u>matrices</u> and live on the <u>links</u> connecting the sites: $\longrightarrow$



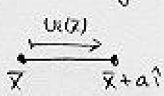
The matrix connecting the sites $\bar{x}$, $\bar{x}+a\hat{\imath}$ may be called $U(\bar{x};\bar{x}+a\hat{\imath})$, or $U_i(\bar{x})$.

We require $U(\bar{x}+a\hat{\imath};\bar{x}) \equiv U^{-1}(\bar{x};\bar{x}+a\hat{\imath})$, so that $U(\bar{x};\bar{x}+a\hat{\imath})\,U(\bar{x}+a\hat{\imath};\bar{x}) = \mathbb{1}$.
This hints at a rôle for $U(\bar{x};\bar{x}+a\hat{\imath})$ as a "parallel transporter": it characterises the movement of information from point $\bar{x}$ to $\bar{x}+a\hat{\imath}$, and if we transport the information back again along the same path, nothing should have changed.
Physically, link matrices correspond to photons, $W^\pm$, Z^0, gluons, mediating information in between matter particles (electrons, quarks, ...).

In order to restrict the class of models of this type further, we will also invoke the property of "<u>gauge-invariance</u>".

Gauge-invariance postulates that physical information is contained only in some subset of all the components of the matrices $U_i(\bar{x})$.
In particular, we can choose a set of matrices $g(\bar{x})$ living on the sites, and require that everything remains the same, if we replace

$$U_i(\bar{x}) \to U_i'(\bar{x}) \equiv g(\bar{x})\,U_i(\bar{x})\,g^{-1}(\bar{x}+a\hat{\imath})$$



(This will be seen to be a generalization of the principle that the phase of a complex wave function is usually not observable in quantum mechanics.)

Given these postulates, we should construct:

(1) an action S which is gauge-invariant.

(2) an integration measure $\int dU_i$ which is gauge-invariant.

(3) show that only gauge-invariant observables have non-vanishing expectation values.

[(4) show that for certain choices of the classes of matrices $U_i(\bar{x})$, $g(\bar{x})$, this system has a non-trivial continuum limit. In fact, it is one of the very few known four-dimensional lattice models which do!]

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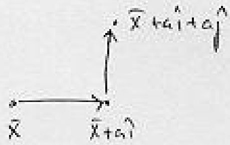
A single matrix, $\begin{cases} U_i(\bar{x}) \to g(\bar{x}) U_i(\bar{x}) g^{-1}(\bar{x}+a\hat{i}) \\ U_i^{-1}(\bar{x}) \to g(\bar{x}+a\hat{i}) U_i^{-1}(\bar{x}) g^{-1}(\bar{x}) \end{cases}$ is not invariant.

A product of two matrices, $U_i(\bar{x}) U_j(\bar{x}+a\hat{i})$ changes as

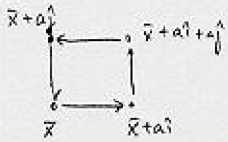
$$U_i(\bar{x}) U_j(\bar{x}+a\hat{i}) \to g(\bar{x}) U_i(x) U_j(\bar{x}+a\hat{i}) g^{-1}(\bar{x}+a\hat{i}+a\hat{j})$$



Therefore, we can obtain an invariant quantity by constructing a <u>closed loop</u>!

$$\Rightarrow \quad P_{ij}(\bar{x}) = U_i(\bar{x}) U_j(\bar{x}+a\hat{i}) U_i^{-1}(\bar{x}+a\hat{j}) U_j^{-1}(\bar{x})$$

$$\to g(\bar{x}) P_{ij}(\bar{x}) g^{-1}(\bar{x})$$



$\Rightarrow \quad \mathrm{Tr}\, P_{ij}(\bar{x})$ is invariant.

P_{ij} is called a plaquette.

To be symmetric, we sum over all i,j. Note: $P_{ji} = P_{ij}^{-1}$!

We know from the previous lecture that the lattice has to be four-dimensional. Indices ranging from 0 (for τ) to 3 (for x_i) are often denoted by μ, ν. And we again introduce a coupling β, and let U_i be $N_c \times N_c$-matrix.

$$\Rightarrow \quad S \equiv \frac{\beta}{2N_c} \cdot \sum_{\bar{x}} \sum_{\mu,\nu=0}^{4} \mathrm{Tr}\left[\overset{\text{conventional}}{\mathbb{1}} - P_{\mu\nu}(\bar{x})\right]$$

Some group theory, and restrictions on $g(\bar{x}), U_i(\bar{x})$

- Unitary matrices, $U(N_c)$: $M = N_c \times N_c$ - matrix, $M^{-1} \equiv M^{+}$.

- Special unitary matrices, $SU(N_c)$: $M^{-1} = M^{+}$, $\det M = 1$.

[recall : $M^{-1} = M^{+} \Rightarrow \det M \det M^{+} = |\det M|^2 = 1$]

- a group G: if $A, B \in G$, then $A \cdot B \in G$.

Both unitary matrices and special unitary matrices form a group:

$$(AB)^{+} = B^{+}A^{+} = B^{-1}A^{-1}$$

$$\Rightarrow (AB)^{+} AB = \mathbb{1} \Rightarrow (AB)^{+} = (AB)^{-1} \qquad \square$$

$$\det(AB) = \det A \det B = +1 \qquad \square$$

$\Rightarrow$ In the following, we assume that $U_i(\bar{x}) \in SU(N_c)$, $g(\bar{x}) \in SU(N_c)$!
Then $U'_i(\bar{x}) = g(x)\, U_i(\bar{x})\, g^{-1}(\bar{x}+a\hat{i}) \in SU(N_c)$, and $P_{ij}(\bar{x}) \in SU(N_c)$.

A special unitary matrix has N_c^2-1 independent components (exercise!)
It can be parametrised as

$$M = \exp\left(i \sum_{a=1}^{N_c^2-1} w^a T^a\right), \quad w^a \in \mathbb{R}, \quad T^a = \text{"generators" of the group}$$

$$M^{+} = \exp\left(-i \sum_{a=1}^{N_c^2-1} w^a T^{a+}\right)$$

$\Rightarrow$ If $(T^a)^{+} = T^a$, then $M^{+} = \exp(-i w^a T^a)$, and $MM^{+} = \mathbb{1} \Rightarrow$ OK

$$\mathrm{Det}\, M = \exp(i w^a \,\mathrm{Tr}\, T^a) = 1 \Rightarrow \mathrm{Tr}\, T^a = 0.$$

Thus the generators of $SU(N_c)$ are Hermitean $(T^{a+} = T^a)$ and traceless matrices. They are often normalised as $\mathrm{Tr}[T^aT^b] = \frac{\delta^{ab}}{2}$.
The structure constants f^{abc} of the group are defined by $[T^a, T^b] \equiv i f^{abc} T^c$.

Group integration

We have now a gauge-invariant action S. In order to define

$$Z = \int \left\{ \prod_{x,i} dU_i(\bar{x}) \right\} \exp(-S) ,$$

we also need to specify the integration measure $\int dU_i(\bar{x})$.

The Haar measure:

Using the parametrization $U_i(w^a) = \exp(i w^a T^a)$, we may expect that $\int dU \sim \int \prod_{a=1}^{N_c^2-1} dw_a$, but this is not quite the full story.

A gauge transformation, $U_i(\bar{x}) \to U_i'(\bar{x}) = g(\bar{x}) U_i(\bar{x}) g^{-1}(\bar{x}+a\hat{i})$, may be viewed as a coordinate transformation $w^a \to (w')^{a}$.

Let us define the object ("metric")

$$g_{ab} \equiv \mathrm{Tr}\left[\frac{\partial U}{\partial w^a} \frac{\partial U^\dagger}{\partial w^b} \right] .$$

In a coordinate transformation, this changes as

$$g'_{a'b'} = \mathrm{Tr}\left[\frac{\partial U}{\partial w'^{a'}} \frac{\partial U^\dagger}{\partial w'^{b'}} \right] = g_{ab} \frac{\partial w^a}{\partial w'^{a'}} \frac{\partial w^b}{\partial w'^{b'}}$$

Consider now the object $\sqrt{\det g} \prod_{a=1}^{N_c^2-1} dw^a$. This is invariant since

$$\sqrt{\det g'} = \sqrt{\det g \cdot \det\left[\partial w^a / \partial w'^{a'} \right]^2} = \sqrt{\det g}\, \left| \det\left[\partial w^a / \partial w'^{a'} \right] \right|$$

$$\prod_a dw'^a = \left| \det\left[\partial w'^a / \partial w^a \right] \right| \prod_a dw^a = \frac{1}{\left| \det\left[\partial w^a / \partial w'^a \right] \right|} \prod_a dw^a$$

Thus: $\int dU_i(\bar{x}) \equiv C \cdot \sqrt{\det g} \int \prod_{a=1}^{N_c^2-1} dw^a$, where the constant C can be chosen so that $\int dU_i(\bar{x}) = 1$.

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Observables

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \left\{ \prod_{\bar{x},i} dU_i(\bar{x}) \right\} \mathcal{O}[U_j(\bar{z})]\, e^{-S[U_k(\bar{y})]}$$

Suppose that $\mathcal{O}$ is not gauge-invariant:

$$U_j(\bar{z}) \to U_j'(\bar{z}) = g(\bar{z}) U_j(\bar{z}) g^{-1}(\bar{z}+a\hat{j}),$$

$$\mathcal{O}[U_j(\bar{z})] \to \mathcal{O}[U_j'(\bar{z})]$$

If $\mathcal{O}$ is a monomial of $U_j(\bar{z})$, for instance, then we might assume for illustration that

$$\mathcal{O} \to g(\bar{z})\, \mathcal{O},$$

if only one $g(\bar{z})$ is chosen non-trivial $(\neq \mathbb{1})$.

Now

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \left\{ \prod_{\bar{x},i} dU_i'(\bar{x}) \right\} \mathcal{O}[U_j'(\bar{z})]\, e^{-S[U_k'(\bar{y})]}$$

$$= \frac{1}{Z} \int \left\{ \prod_{\bar{x},i} dU_i(\bar{x}) \right\} g(\bar{z})\, \mathcal{O}[U_j(\bar{z})]\, e^{-S[U_k(\bar{y})]}$$ (using the invariance of S and integration measure)

$$= g(\bar{z}) \langle \mathcal{O} \rangle.$$

What kind of elements are there in $\{g(\bar{z})\}$?

For instance, roots of unity, $g = e^{\frac{i2\pi n}{N_c}} \cdot \mathbb{1}_{N_c \times N_c}$, $n = 0, \ldots, N_c - 1$.

[This is called the center $Z(N_c)$ of $SU(N_c)$.]

$$g^\dagger = g^{-1} \quad \& \quad \det g = e^{\frac{i2\pi n}{N_c} \cdot N_c} = 1.$$

Thus $$\langle \mathcal{O} \rangle = e^{\frac{i2\pi n}{N_c}} \langle \mathcal{O} \rangle = \frac{1}{N_c} \sum_{n=0}^{N_c-1} e^{\frac{i2\pi n}{N_c}} \langle \mathcal{O} \rangle = 0!$$

$\Rightarrow$ In general only gauge-invariant observables

[Tr P_{ij}, Tr $\square$, Tr ⌐□, etc]

have non-trivial expectation values.