Euclidean vs Minkowskian spacetime

Quantum statistical mechanics

So far the physical picture we had in mind was that of classical statistical mechanics:

$$S = \beta H \quad , \quad Z = \int \mathcal{D}\phi \, e^{-S} \quad , \quad \beta \propto \frac{1}{k_B \cdot T} \quad , \quad \text{space-dimensionality} = 3 .$$

What would the pattern be in quantum statistical mechanics?

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle \quad , \qquad (*)$$

where $\hat{H}$ is the Hamiltonian and $|n\rangle$ an energy eigenstate $\Rightarrow$ sum over discrete E_n !

A more useful representation is obtained in configuration space. Consider a single particle. Functional integral:

$$\langle x_b | e^{-\frac{i\hat{H}t}{\hbar}} | x_a \rangle = \int_{x(0)=x_a}^{x(t)=x_b} \mathcal{D}x \; e^{\frac{i}{\hbar}\int_0^t dt' \, \mathcal{L}} \quad ,$$

$$\mathcal{L} = \frac{m}{2}(\dot{x})^2 - U(x) \quad , \quad U(x) = \text{potential} \quad , \quad \dot{x} = \frac{dx}{dt}$$

Functional integral is a sum over all trajectories with $x(0) = x_a$, $x(t) = x_b$.

Employ now that $\mathbb{1} = \sum_n |n\rangle\langle n| = \int_{-\infty}^{\infty} dx \, |x\rangle\langle x|$, and identify $it' \equiv \tau$, $it \equiv \beta\hbar$

$$\Rightarrow \quad Z = \int_{-\infty}^{\infty} dx \, \langle x | e^{-\beta \hat{H}} | x \rangle$$

$$= \int_{x(0)=x(\beta\hbar)} \mathcal{D}x \; e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left\{ \frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 + U(x) \right\}}$$

Thus we have obtained a path integral representation for the partition function Z, without the need to solve for energy eigenvalues as in equation $(*)$! Classical $\beta(\dots) \rightarrow$ quantum $\frac{1}{\hbar}\int_0^{\beta\hbar}(\dots)$!

Statistical quantum field theory

The previous example was for one particle. How about a quantum field theory?

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\,; \qquad V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

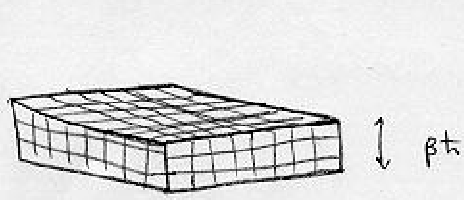
$$= \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\sum_{i=1}^{3}(\partial_i\phi)^2 - V(\phi)\,; \qquad \partial_i = \frac{\partial}{\partial x_i}$$

It's like one particle, with

$$m \to 1$$
$$x \to \phi$$
$$\frac{d}{dt} \to \frac{\partial}{\partial t}$$
$$U(x) \to \frac{1}{2}\sum_{i=1}^{3}(\partial_i\phi)^2 + V(\phi)$$

$$\Rightarrow \quad Z = \int\limits_{\phi(0,\bar{x}) = \phi(\beta\hbar,\bar{x})} \mathcal{D}\phi \; e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \sum_{i=1}^{3}\frac{1}{2}\left(\frac{\partial\phi}{\partial x_i}\right)^2 + V(\phi)\right\}}$$

$\Rightarrow$ To treat a three-dimensional system quantum-mechanically, need to carry out a four-dimensional functional integral, with periodic boundary conditions over the τ-direction!



How to recover the classical limit? For $\hbar \to 0$, the τ-direction is very narrow. For configurations which do not depend on τ,

$$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x \left\{ \dots \right\} \approx \beta \cdot \int d^3x \left\{ \sum_{i=1}^{3}\frac{1}{2}\left(\frac{\partial\phi}{\partial x_i}\right)^2 + V(\phi)\right\}$$

Correlation functions

Operators in the Heisenberg-picture:

$$\hat{O}(t) \equiv e^{\frac{i\hat{H}t}{\hbar}} \hat{O}(0) e^{-\frac{i\hat{H}t}{\hbar}} .$$

The simplest correlation functions:

- one-point function:

$$\langle \hat{O}(t) \rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta \hat{H}} e^{\frac{i\hat{H}t}{\hbar}} \hat{O}(0) e^{-\frac{i\hat{H}t}{\hbar}} \right] = \frac{1}{Z} \mathrm{Tr}\, e^{-\beta \hat{H}} \hat{O}(0) .$$

- two-point "equal-time" functions:

$$\langle \hat{A}(t)\, \hat{B}(t) \rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta \hat{H}} e^{\frac{i\hat{H}t}{\hbar}} \hat{A}(0) e^{-\frac{i\hat{H}t}{\hbar}} e^{\frac{i\hat{H}t}{\hbar}} \hat{B}(0) e^{-\frac{i\hat{H}t}{\hbar}} \right] = \langle \hat{A}(0)\, \hat{B}(0) \rangle .$$

For these cases, a path integral expression is easily obtained:

$$\langle \hat{A}(0)\, \hat{B}(0) \rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{A}(0)\, \hat{B}(0) \right]$$

$$= \lim_{M\to\infty} \frac{1}{Z} \mathrm{Tr}\left[\overbrace{e^{-\beta\frac{\hat{H}}{M}}\; e^{-\beta\frac{\hat{H}}{M}} \cdots\; e^{-\beta\frac{\hat{H}}{M}}}^{M \text{ copies}}\; \hat{A}(0)\; \hat{B}(0) \right]$$

$$\mathbb{1} = \int d\phi_M |\phi_M\rangle\langle\phi_M| \qquad \mathbb{1} = \int d\phi_1 |\phi_1\rangle\langle\phi_1| \qquad \mathbb{1} = \int d\phi_0 |\phi_0\rangle\langle\phi_0|$$

$$\hat{B}(\hat{\phi})|\phi_0\rangle = \hat{B}(\phi_0)|\phi_0\rangle \equiv B(\phi_0)|\phi_0\rangle$$

So it goes through just like for the partition function itself.

$$\Rightarrow \frac{1}{Z} \int_{\phi(0,\vec{x}) = \phi(\beta\hbar,\vec{x})} \mathcal{D}\phi \quad A(\phi(0,\vec{x}))\, B(\phi(0,\vec{x}))\, \exp\left\{ -\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int d^3x \left[\ldots \right] \right\} .$$

Therefore this kind of expectation values can be obtained directly from the "Euclidean" functional integral with a fourth coordinate τ.

The situation is more complicated for <u>non-equal time correlation functions</u>.

Let us define:

$$\Pi^<(q_0) \equiv \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \langle \hat{O}(0)\, \hat{O}(t) \rangle$$

$$\Pi^>(q_0) \equiv \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \langle \hat{O}(t)\, \hat{O}(0) \rangle$$

$$\rho(q_0) \equiv \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \left\langle \tfrac{1}{2} \left[\hat{O}(t), \hat{O}(0) \right] \right\rangle$$

$$\Delta(q_0) \equiv \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \left\langle \tfrac{1}{2} \left\{ \hat{O}(t), \hat{O}(0) \right\} \right\rangle .$$

Only one of these is independent; usually one chooses the <u>spectral density $\rho(q_0)$</u>.

Insert twice $1 = \sum_n |n\rangle\langle n|$:

$$\Pi^<(q_0) = \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \frac{1}{Z} \operatorname{Tr}\left[e^{-\beta \hat{H}}\, \hat{O}(0)\, \underset{\uparrow\, 1}{e^{\frac{i\hat{H}t}{\hbar}}}\, \hat{O}(0)\, \underset{\uparrow\, 1}{e^{-\frac{i\hat{H}t}{\hbar}}} \right]$$

$$= \int_{-\infty}^{\infty} dt\, e^{iq_0 t} \frac{1}{Z} \sum_{n,m} e^{-\left(\beta + \frac{it}{\hbar}\right) E_m} e^{\frac{it}{\hbar} E_n} \langle m|\hat{O}|n\rangle \langle n|\hat{O}|m\rangle$$

$$= \frac{1}{Z} \sum_{n,m} e^{-\beta E_m}\, 2\pi\, \delta\left(q_0 + \frac{E_n - E_m}{\hbar} \right) \langle m|\hat{O}|n\rangle \langle n|\hat{O}|m\rangle$$

$$\Pi^>(q_0) = \frac{1}{Z} \sum_{n,m} e^{-\beta E_n}\, 2\pi\, \delta\left(q_0 + \frac{E_n - E_m}{\hbar} \right) \langle m|\hat{O}|n\rangle \langle n|\hat{O}|m\rangle$$

$\nwarrow E_n = E_m - q_0\hbar$

$$= e^{\beta q_0 \hbar}\, \Pi^<(q_0)$$

$$\rho(q_0) = \frac{1}{2}\left[\Pi^> - \Pi^< \right] = \frac{1}{2}\left(e^{\beta q_0 \hbar} - 1 \right) \Pi^<(q_0)$$

$$\Rightarrow \quad \Pi^<(q_0) = 2\, n_b(\hbar q_0)\, \rho(q_0)$$

$$\Pi^>(q_0) = 2\, e^{\beta \hbar q_0}\, n_b(\hbar q_0)\, \rho(q_0)$$

$$\Delta(q_0) = \left(1 + 2 n_b(\hbar q_0) \right) \rho(q_0) \quad ; \quad n_b(\omega) = \frac{1}{e^{\beta\omega} - 1}$$

It can be shown [we may return to this later] that:

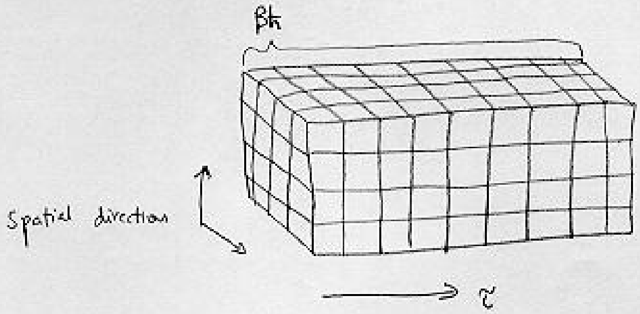
$$\Pi(\tau) \equiv \frac{1}{Z} \int_{\phi(0,\bar{x}) = \phi(\beta\hbar, \bar{x})} \mathcal{D}\phi \quad O(\tau)\, O(0)\; e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^dx \left\{ \frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}\sum_i (\partial_i \phi)^2 + V(\phi) \right\}}$$

$$= \int_0^{\infty} \frac{dq_0}{\pi}\, \rho(q_0)\, \frac{\cosh\left(\frac{\beta\hbar}{2} - \tau \right) q_0}{\sinh \frac{\beta\hbar q_0}{2}} \quad ; \quad O(\tau) \equiv O[\phi(\tau)] .$$

If this could be inverted for $\rho(q_0)$, real-time (Minkowskian) correlators

The limit of zero temperature, or "vacuum"

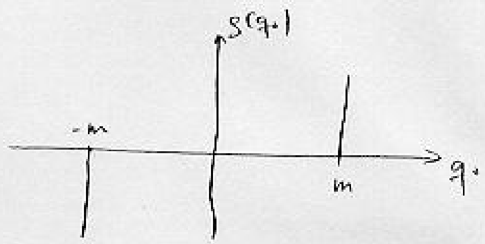
Take now a different type of a box:



For $\beta\hbar \to \infty$,
$$\frac{\cosh\left(\frac{\beta\hbar}{2}-\tau\right)q_0}{\sinh\frac{\beta\hbar q_0}{2}} = \frac{e^{\left(\frac{\beta\hbar}{2}-\tau\right)q_0} + \cancel{e^{-\left(\frac{\beta\hbar}{2}-\tau\right)q_0}}}{e^{\frac{\beta\hbar}{2}q_0} - \cancel{e^{-\frac{\beta\hbar}{2}q_0}}} \approx e^{-\tau q_0}$$

$$\Rightarrow \quad \Pi(\tau) = \int_0^\infty \frac{dq_0}{\pi}\, \rho(q_0)\, e^{-\tau q_0} .$$

For a free particle of mass m, $\rho(q_0) = C\left[\delta(m-q_0) - \delta(m+q_0)\right]$



$$\Rightarrow \quad \Pi(\tau) = \frac{C}{\pi} e^{-m\tau} \quad \Rightarrow$$

mass at $T=0$ can be determined from the exponential falloff of a Euclidean correlator!

This corresponds to <u>the Wick rotation</u> of usual quantum field theory, where the behaviour (for a particle at rest) would be $\sim e^{-imt}$ in Minkowski space.

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