Approach to the continuum limit

$$S = \sum_{\bar{x}} a^d \left[\frac{1}{2a^2} \sum_{i=1}^{d} \left(\phi(\bar{x}+a\hat{i}) - \phi(\bar{x}) \right)^2 + \frac{1}{2} m^2 \phi^2(\bar{x}) + \frac{1}{4} \lambda \phi^4(\bar{x}) \right]$$

The problem: as we have seen, this system can be treated with various perturbative techniques (in some parts of the parameter space) and non-perturbatively with lattice Monte Carlo simulations (for any $\lambda \geq 0$ parameter values). But what happens in the continuum limit $a \to 0$?

(a) Does the limit exist?

(b) If yes, is it "non-trivial"?

To be as unrestrictive as possible, we in fact ask these questions only for a certain subclass of observables, called the renormalised observables. Some other objects, called the bare quantities, are allowed to vary in which ever way may help us to reach $a \to 0$.

Bare quantities: from now on, let us denote

$$\phi \equiv \phi_0 \quad , \quad m^2 \equiv m_0^2 \quad , \quad \lambda \equiv \lambda_0$$

These are called the bare field and the bare couplings.

Renormalised quantities: there is no unique choice for the renormalised quantities. One could choose physical observables, like the correlation length ξ. Here we rather define them as follows:

- $$\sum_{\bar{x}} a^d \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle e^{-i\bar{p}\cdot\bar{x}} \equiv \frac{Z_R}{m_R^2 + \bar{p}^2 + \mathcal{O}(\bar{p}^4)}$$

Thus, if

$$\chi_2 \equiv \sum_{\bar{x}} a^d \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle$$

$$\mu_2 \equiv \sum_{\bar{x}} a^d \bar{x}^2 \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle ,$$

then

$$m_R^2 = \frac{\chi_2}{\mu_2}$$

$$Z_R = \frac{\chi_2^2}{\mu_2}$$

- To define a renormalised λ_R, consider

$$\chi_4 \equiv \sum_{\bar{x}} a^d \sum_{\bar{y}} a^d \sum_{\bar{z}} a^d \left\langle \phi_0(\bar{x})\phi_0(\bar{y})\phi_0(\bar{z})\phi_0(0) \right\rangle .$$

Then

$$\lambda_R \equiv -\frac{1}{6}\cdot\frac{1}{\mu_2^2}\cdot\left(\chi_4 - 3V\chi_2^2 \right) .$$

The subtraction removes the term given by the Wick contraction, i.e., a contribution existing even for $\lambda_0 = 0$.

$\Rightarrow$ Other choices of Z_R, m_R^2, λ_R are possible, and correspond to other renormalization schemes. [ski:ms]

Formulation of question in terms of m_0^2, λ_0 & m_R^2, λ_R :

Can the dimensionless parameters $a^2 m_0^2(a)$, $a^{d-4}\lambda_0(a)$ be tuned so that

(a) $\quad a \cdot m_R \to 0 \qquad$ ($\equiv$ continuum limit)

(b) $\quad \dfrac{\lambda_R}{m_R^{d-4}} \neq 0 \qquad$ ($\equiv$ non-trivial theory).

[In a trivial theory, $\lambda_R \to 0$. Then the system is Gaussian in the continuum limit: no interactions, just free particles. This is not the world we observe!]

General remarks considering continuum limit:

(1) $\frac{\xi}{a} \approx \frac{1}{m_R a} \longrightarrow \infty$

$\Rightarrow$ a necessary requirement is the existence of a continuous (2nd order) phase transition in the lattice model. Recall that the lattice model is parametrised only by $\kappa, \hat{\lambda}$ (p. 26).

(2) Universality:

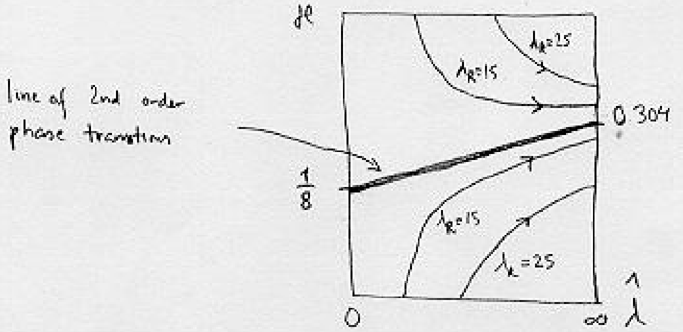
Since the critical properties of 2nd order phase transitions are universal, the same continuum limit is reached for many different models, e.g. Ising and $\phi \in \mathbb{R}$!

(3) The (apparent) problem of divergences: in general, $m_0^2, \lambda_0 \to \infty$ when the continuum limit is approached, in order for m_R^2, λ_R to stay finite. It took decades for physicists to accept this.

(4) Renormalizability means: when any physical quantity $\mathcal{O}$ is expressed in terms of m_R^2, λ_R (rather than m_0^2, λ_0), the expression is finite in the continuum limit:

$$\lim_{a m_R \to 0} \mathcal{O}(m_R^2, \lambda_R) \ \exists \ !$$

(5) There are various types of renormalization group equations. An example:



On each curve, λ_R is fixed. The value of $a m_R$ decreases along the curve. There is a Callan-Symanzik equation of the form

$$m_R \left.\frac{\partial \kappa}{\partial m_R}\right|_{\hat{\lambda}} = \beta_1 \cdot \lambda_R^2 + \beta_2 \cdot \lambda_R^3 + \dots$$

M. Lüscher & P. Weisz,
Nucl. Phys. B 290 (1987) 25
295 (1988) 65
300 (1988) 325
31[illegible] (1989) 705

An example: what can be learned about continuum limit in weak coupling expansion?

(a) p. 24 : $$\langle \phi_0(\tilde{x})\phi_0(\tilde{y})\rangle = \int_{\tilde{p}} e^{i\tilde{p}\cdot(\tilde{x}-\tilde{y})} \frac{1}{\tilde{p}^2+m_0^2+3\lambda_0 \int_{\tilde{q}} \frac{1}{\tilde{q}^2+m_0^2}} + O(\lambda_0^2)$$

$$\Rightarrow \quad Z_R = 1 + O(\lambda_0^2)$$

$$m_R^2 = m_0^2 + 3\lambda_0 \int_{\tilde{q}} \frac{1}{\tilde{q}^2+m_0^2}$$

(b) Let us compute λ_R from p. 57 !

$$\exp(-S_I) = 1 - S_I + \frac{1}{2} S_I^2$$

χ_4 $O(\lambda_0^0)$:  $\Rightarrow$ subtracted by χ_2^2

$O(\lambda_0^1)$:  $\Rightarrow$ subtracted by χ_2^2

 $\Rightarrow$ (b1)

$O(\lambda_0^2)$  $\Rightarrow$ (b2)

$$(b1): \int_{x,y,z} \int_{p,q,r,s} e^{ip\cdot x} e^{iq\cdot y} e^{iz\cdot r} \langle \phi_0(p)\phi_0(q)\phi_0(r)\phi_0(s) \left(-\frac{\lambda_0}{4}\right) \int_w \phi_0^4(w) \rangle$$

4 · 3 · 2

$$= -6\lambda_0 \cdot \left(\frac{1}{m_0^2}\right)^4$$

$$(b2): \int_{x,y,z} \int_{p,q,r,s} e^{ip\cdot x} e^{iq\cdot y} e^{iz\cdot r} \langle \phi_0(p)\phi_0(q)\phi_0(r)\phi_0(s)\ \frac{1}{2}\frac{\lambda_0^2}{16} \int_w \phi_0\phi_0\phi_0\phi_0 \int_v \phi_0\phi_0\phi_0\phi_0 \rangle$$

8 ; 3 × 4 × 3 × 2 ; 4 × 6 × 3 × 2

$$\frac{8 \times 9 \times 24}{2 \times 16} = 6 \times 9$$

$$= + 6 \cdot \lambda_0^2 \cdot 9 \cdot \left(\frac{1}{m_0^2}\right)^4 \cdot \int_{\tilde{q}} \frac{1}{(\tilde{q}^2+m_0^2)^2}$$

$$\Rightarrow \quad \lambda_R = \lambda_0 - 9\lambda_0^2 \int_{\tilde{q}} \frac{1}{(\tilde{q}^2+m_0^2)^2} + O(\lambda_0^3) .$$

What this implies on the continuum limit depends on the dimension :

$d=3$: $\int_q \frac{1}{(\tilde{q}^2+m_R^2)^2} = \frac{1}{8\pi m_R}\left[1+\mathcal{O}(am_R)\right]$ (exercise)

$\Rightarrow \quad \frac{\partial}{\partial a}\lambda_R\Big|_{\lambda_0} = \mathcal{O}\left(\frac{\lambda_0^2}{8\pi}\right) = \mathcal{O}\left(\frac{\lambda_R^2}{8\pi}\right)$

$\Rightarrow \quad \lambda_R = \lambda_R^{(a=0)} + \mathcal{O}\left(\frac{\lambda_R^2}{8\pi}\right) a + \mathcal{O}(a^2)$

$\Rightarrow \quad \lambda_R$ can be finite as $a\to 0$

$\Rightarrow$ a non-trivial continuum limit exists.

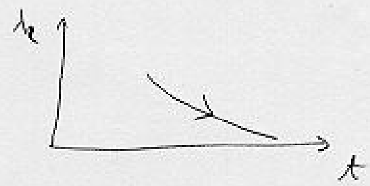
$d=4$: $\int_q \frac{1}{(\tilde{q}^2+m_R^2)^2} = \frac{1}{(4\pi)^2}\left[\ln\frac{1}{(am_R)^2} + \text{const} + \mathcal{O}(am_R)^2\right]$ (exercise)

$\Rightarrow \quad a\frac{\partial}{\partial a}\lambda_R\Big|_{\lambda_0} = +\lambda_0^2\frac{9}{8\pi^2} + \mathcal{O}(\lambda_0^3) = +\lambda_R^2\frac{9}{8\pi^2} + \mathcal{O}(\lambda_R^3)$

Let $t = \ln\frac{1}{a}$; $t\to\infty$ as $a\to 0$

$\Rightarrow \quad a\frac{\partial}{\partial a} = -\partial_t$

$\Rightarrow \quad \partial_t\lambda_R = -\lambda_R^2\frac{9}{8\pi^2} + \mathcal{O}(\lambda_R^3)$



$\Rightarrow \quad \lambda_R(t\to\infty) = 0$

$\Rightarrow$ In $d=4$ the continuum limit may exist but the theory is trivial then !

Summary : approach to the continuum limit

Action

Field theory notation :

- bare field $\phi_0(\bar{x})$
- bare parameters m_0^2, λ_0
- lattice spacing a

Spin model notation :

- (bare) field $s(\bar{x})$
- parameters κ, $\hat{\lambda}$, both of which are dimensionless

Observables

- $\chi_2 = \sum_{\bar{x}} a^d \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle$

  $\chi_4 = \sum_{\bar{x},\bar{y},\bar{z}} a^{3d} \langle \phi_0(\bar{x}) \phi_0(\bar{y}) \phi_0(\bar{z}) \phi_0(\bar{0}) \rangle$

  ⋮

- $\mu_2 = \sum_{\bar{x}} a^d \, \bar{x}^2 \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle$

  ⋮

- $\xi^{-1} = - \lim_{|\bar{x}| \to \infty} \frac{1}{|\bar{x}|} \ln \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle$

  ⋮

<u>And arbitrarily many others.</u>

Renormalised parameters ≈ physical observables

- For each bare field, there is a wave function renormalization factor Z_R.
- For each bare parameter, there is a renormalised parameter: m_R^2, λ_R
- The definitions of Z_R, m_R^2, λ_R are not unique (they depend on the "scheme"), but for instance one could choose

$$Z_R \equiv \frac{\chi_2^2}{\mu_2}$$

$$m_R^2 \equiv \frac{\chi_2}{\mu_2}$$

(So that

$$\sum_{\bar{x}} a^d \langle \phi_0(\bar{x}) \phi_0(\bar{0}) \rangle e^{-i\bar{p}\cdot\bar{x}} \equiv \frac{Z_R}{m_R^2 + \bar{p}^2 + \mathcal{O}(\bar{p}^4)} \quad)$$

$$\lambda_R \equiv -\frac{1}{6\mu_2^2} \left(\chi_4 - 3V\chi_2^2 \right)$$

Weak coupling expansion

- With these definitions,

$$Z_R = 1 + \mathcal{O}(\lambda_0^2)$$

$$m_R^2 = m_0^2 + 3\lambda_0 \int_{\tilde{q}} \frac{1}{\tilde{q}^2 + m_0^2} + \mathcal{O}(\lambda_0^2)$$

(p. 24)

$$\lambda_R = \lambda_0 - 9\lambda_0^2 \int_{\tilde{q}} \frac{1}{(\tilde{q}^2+m_0^2)^2} + \mathcal{O}(\lambda_0^3)$$

$$(\times + \asymp)$$

What happens when a→0

- If bare parameters can b[e] tuned such that am_R – continuum limit exists.
- If $\frac{\lambda_R}{m_R^{d-4}} \neq 0$, it is called "non-trivial".
- In the lattice model, $m_R a \propto \frac{a}{\xi} \to 0$ only [if] there is a <u>2nd order</u> phase transition.
- Problem of <u>divergences</u>: in order for m_R^2, λ_R to st[ay] finite as $a \to 0$, m_0^2, λ_0 d[…]
- <u>Renormalizibility</u>: when [a] "physical observable" is e[xpressed] in terms of m_R^2, λ_R, rat[her] than m_0^2, λ_0, the expr[ession] <u>remains finite</u> for a→[0]
- Renormalization group:

$$m_R \frac{\partial \lambda_R}{\partial m_R} \Big|_{\lambda_0} = \beta_1 \lambda_R^2 + \beta_2 \lambda_R^3 \ldots$$

(and other variants).