"Strong coupling expansion" — hopping parameter expansion

(or high-temperature expansion = small-β expansion)

This class of expansions comes from the side of spin models, rather than field theory, and requires that we go back into that notation.

P. 12:

$$\sum_{\bar{x}} a^d \left\{ \frac{1}{2a^2} \sum_{i=1}^{d} \left[\phi(\bar{x}+a\hat{i}) - \phi(\bar{x})\right]^2 + \frac{1}{2} m^2 \phi^2(\bar{x}) + \frac{1}{4} \lambda \phi^4(\bar{x}) \right\}$$

$\phi = \sqrt{2\kappa a^{2-d}}\, s$

$$= \sum_{\bar{x}} \left\{ -2\kappa \sum_{i=1}^{d} s(\bar{x})\, s(\bar{x}+a\hat{i}) + (2d + m^2a^2)\kappa\, s^2(\bar{x}) + \lambda a^{4-d} \kappa^2 s^4(\bar{x}) \right\}$$

So far κ is arbitrary. Let us now express it in terms of the old parameters by requiring

$$(2d + m^2a^2)\kappa \equiv 1 - 2\hat{\lambda}$$
$$\hat{\lambda} \equiv \lambda a^{4-d} \kappa^2$$

$$\Rightarrow \quad S = \sum_{\bar{x}} \left\{ -2\kappa \sum_{i=1}^{d} s(\bar{x})\, s(\bar{x}+a\hat{i}) + s^2(\bar{x}) + \hat{\lambda}\left(s^2(\bar{x}) - 1\right)^2 \right\}$$

The hopping parameter expansion is an expansion in $\kappa \ll 1$, for fixed $\hat{\lambda}$. Since in spin models $\beta = 2\kappa$, this is also called the "high-temperature expansion". Closely related is the "linked cluster expansion". Why this could also be called the "strong coupling expansion" becomes clear later on, but at least one may note that any value of $\hat{\lambda}$ is allowed.

NB. What is the typical order of magnitude of κ?

Assume that we approach the "continuum limit" of finer and finer lattices. Then $m^2a^2 \to 0$

$$\Rightarrow \quad \kappa \to \frac{1}{2d} - \frac{\hat{\lambda}}{d} \xrightarrow{d<4} \frac{1}{2d} \ll 1 .$$

The procedure is again straightforward.

Let us introduce

$$d\mu(s(\bar{x})) \equiv ds(x)\, e^{-s^2(x) - \hat{\lambda}(s^2(\bar{x}) - 1)^2} \quad , \quad \hat{\lambda} \geq 0 .$$

(1) Compute partition function to first non-trivial order

$O(\kappa^0)$:

$$Z_0 = \prod_{\bar{x}} \int ds(\bar{x})\, e^{-S}$$

$$= \left[\int ds(\bar{x})\, e^{-s^2(\bar{x}) - \hat{\lambda}(s^2(\bar{x})-1)^2} \right]^{\left(\sum_{\bar{x}}\right)} ,$$

where $\sum_{\bar{x}}$ = number of lattice sites = N_{tot} = $N_1 N_2 \cdots N_d$.

The integral extrapolates between $\sqrt{\pi}$ $(\hat{\lambda} = 0)$

and $\sqrt{\pi} \cdot \frac{e^{-1}}{\sqrt{\hat{\lambda}}}$ $(\hat{\lambda} \to \infty)$.

$O(\kappa^1)$:

$$Z_1 = \prod_{\bar{x}} \int d\mu(s(\bar{x})) \cdot 2\kappa \sum_{\bar{z}} \sum_{i=1}^{d} s(\bar{z})\, s(\bar{z} + a\hat{i}) = 0 .$$

$O(\kappa^2)$:

$$Z_2 = \prod_{\bar{x}} \int d\mu(s(\bar{x})) \cdot 2 \cdot \kappa^2 \sum_{\bar{y}} \sum_{\bar{z}} \sum_{i=1}^{d} \sum_{j=1}^{d} s(\bar{y})\, s(\bar{y}+a\hat{i})\, s(\bar{z})\, s(\bar{z}+a\hat{j})$$

Because each $d\mu(s(\bar{x}))$ is symmetric in $s \to -s$, only the term with $\bar{y} = \bar{z}$, $i = j$ contributes!

$$\Rightarrow Z_2 = 2\kappa^2 \sum_{\bar{y}} \sum_{i=1}^{d} \left\{ \prod_{\bar{x}} \int d\mu(s(\bar{x})) \right\} \cdot \frac{\int d\mu(s(\bar{y}))\, s^2(\bar{y})}{\int d\mu(s(\bar{y}))} \cdot \frac{\int d\mu(s(\bar{y}+a\hat{i}))\, s^2(\bar{y}+a\hat{i})}{\int d\mu(s(\bar{y}+a\hat{i}))}$$

$$= 2\kappa^2 \cdot N_{tot} \cdot d \cdot Z_0 \cdot \left(\langle s^2 \rangle_1 \right)^2 ,$$

where $\langle \dots \rangle_1 \equiv \dfrac{\int d\mu(s)\, (\dots)}{\int d\mu(s)}$.

Thus: $$\frac{Z}{Z_0} = 1 + 2\,d\,N_{tot}\,\kappa^2 \left(\langle s^2\rangle_1\right)^2 + \mathcal{O}(\kappa^4).$$

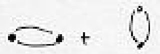
The integral remaining: $$\langle s^2\rangle_1 = \begin{cases} \frac{1}{2} & (\hat\lambda = 0) \\ 1 & (\hat\lambda = \infty) \end{cases}$$

Graphical representation:

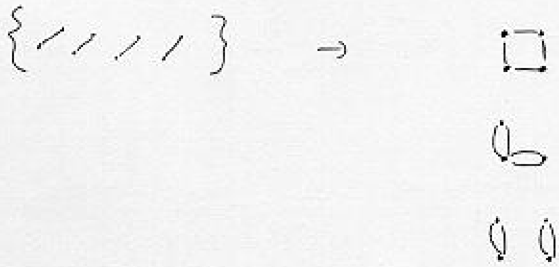
$2\kappa\, s(\bar x)\, s(\bar x + a\hat\imath)$ =  = "hop" from site $\bar x$ to $\bar x + a\hat\imath$

The "rule" is now that an even number of sites must overlap $\Rightarrow$ closed loops!

$\mathcal{O}(\kappa)$: $= 0$

$\mathcal{O}(\kappa^2)$:  $\propto d \cdot \left(\langle s^2\rangle_1\right)^2$

$\mathcal{O}(\kappa^4)$:



etc

The expansion has been worked out to 14th order in M. Lüscher & P. Weisz, Nucl. Phys. B 290 (1987) 25, and in references therein.

(2) <u>As another example, let us study the propagator</u>

$$G(\bar{x},\bar{y}) \equiv \langle s(\bar{x})\, s(\bar{y}) \rangle$$

The propagator can be computed to all orders in $\varkappa$ by using the "random walk approximation": treat all expectation values $\langle s^n \rangle_1$ as if they were Gaussian ($\Rightarrow$ Wick's theorem).

Introduce a hopping matrix $H_{\bar{u}\bar{v}} \equiv \sum_{\hat{\imath}=1}^{d} \left(\delta_{\bar{u}+\hat{\imath},\bar{v}} + \delta_{\bar{v}+\hat{\imath},\bar{u}} \right)$

It is non-zero if and only if $\bar{u},\bar{v}$ are nearest neighbours.

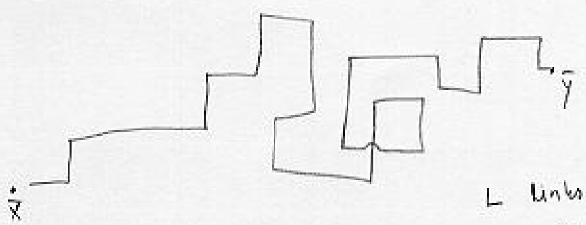
$$\Rightarrow \quad \langle s(\bar{x})\, s(\bar{y}) \rangle = \frac{1}{Z} \int \left\{ \prod_{\bar{z}} d\mu(s(\bar{z})) \right\} s(\bar{x}) s(\bar{y}) \exp\left(\varkappa \sum_{\bar{u}\bar{v}} s(\bar{u}) H_{\bar{u}\bar{v}} s(\bar{v}) \right)$$

L-th order:

$$\frac{1}{L!} \varkappa^L \left\langle s_{\bar{x}}\, s_{\bar{y}} \quad s_{\bar{u}_1} H_{\bar{u}_1\bar{v}_1} s_{\bar{v}_1} \quad s_{\bar{u}_2} H_{\bar{u}_2\bar{v}_2} s_{\bar{v}_2} \quad \cdots \quad s_{\bar{u}_L} H_{\bar{u}_L\bar{v}_L} s_{\bar{v}_L} \right\rangle$$

2L … 2(L-1)

$$= (2\varkappa)^L \left[\langle s^2 \rangle_1 \right]^{L+1} \left[H^L \right]_{\bar{x}\bar{y}}$$



L links,
L+1 sites

Note that here we picked up only connected contractions, since disconnected are cancelled by corrections to Z, like for the weak coupling expansion.

Go to momentum space. Since we have scaled "a" away, put $a \to 1$ in Fourier transformations.

$$H_{\bar{p}\bar{q}} = \sum_{\bar{x},\bar{y}} e^{i\bar{p}\cdot\bar{x}} e^{i\bar{q}\cdot\bar{y}} \sum_{i=1}^{d} \left(\delta_{\bar{x}+\hat{i},\bar{y}} + \delta_{\bar{x}-\hat{i},\bar{y}} \right)$$

$$= \sum_{\bar{x}} e^{i\bar{p}\cdot\bar{x}} \sum_{i=1}^{d} \left(e^{i\bar{q}\cdot(\bar{x}+\hat{i})} + e^{i\bar{q}\cdot(\bar{x}-\hat{i})} \right)$$

$$= \delta_{\bar{p}+\bar{q}} \sum_{i=1}^{d} \left(e^{ip_i} + e^{-ip_i} \right) = \delta_{\bar{p}+\bar{q}} H(\bar{p})$$

In Fourier-space, the whole correlator becomes

$$G(\bar{x},\bar{y}) = \int_p \sum_{L=0}^{\infty} (2\varkappa)^L \left[\langle s^2 \rangle_1 \right]^{L+1} \left[H(\bar{p}) \right]^L e^{i\bar{p}\cdot(\bar{x}-\bar{y})}$$

$$= \int_p e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{\langle s^2 \rangle_1}{1 - 2\varkappa \langle s^2 \rangle_1 H(\bar{p})}$$

Cross check :

$$* \quad H(\bar{p}) = \sum_{i=1}^{d} \left(e^{ip_i} + e^{-ip_i} \right) = 2d + \sum_{i=1}^{d} \left(e^{i\frac{p_i}{2}} - e^{-i\frac{p_i}{2}} \right)^2$$

$$= 2d - \tilde{p}^2 \quad , \quad \tilde{p}^2 = \sum_{i=1}^{d} \tilde{p}_i^2 , \quad \tilde{p}_i = 2 \sin\frac{p_i}{2} = \frac{2}{a} \sin\frac{a p_i}{2} \Big|_{a=1}$$

$$* \quad \text{In the limit } \hat{\lambda} = 0, \quad \langle s^2 \rangle_1 = \frac{1}{2} .$$

$$* \quad (2d + m^2 a^2)\varkappa = 1 - 2\hat{\lambda}$$

$$\Rightarrow \text{for } \hat{\lambda} = 0, \quad G(\bar{x},\bar{y}) = \int_p e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{\frac{1}{2}}{1 - \varkappa (2d - \tilde{p}^2)}$$

$$= \frac{1}{2\varkappa} \int_p e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{\tilde{p}^2 + m^2 a^2}$$

Like in weak coupling !

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