An example of a 1-loop computation

Let us compute the correlation length for the case $\phi \in \mathbb{R}$, $d=3$, as a function of the parameters m^2, λ. We assume $m^2 > 0$!

$$S = \sum_{\bar{x}} a^d \left[\frac{1}{2} \sum_{i=1}^{d} \frac{|\phi(\bar{x}+a\hat{i}) - \phi(\bar{x})|^2}{a^2} + \frac{1}{2} m^2 \phi^2(\bar{x}) + \frac{1}{4} \lambda \phi^4(\bar{x}) \right]$$

$$\phi(\bar{x}) = \int_p e^{i\bar{p}\cdot\bar{x}} \phi(\bar{p})$$

$$\langle \phi(\bar{p}) \phi(\bar{q}) \rangle_0 = V \cdot \delta_{\bar{p}+\bar{q},0 \bmod \frac{2\pi}{a}} \frac{1}{\sum_{i=1}^{d} \tilde{p}_i^2 + m^2}, \qquad \tilde{p}_i = \frac{2}{a} \sin\left(\frac{a p_i}{2}\right), \quad \tilde{p}^2 \equiv \sum_{i=1}^{d} \tilde{p}_i^2$$

Definition of correlation length ξ (p. 4):

$$\lim_{|\bar{x}-\bar{y}| \to \infty} \langle \phi(\bar{x}) \phi(\bar{y}) \rangle = \text{Polynomial}(|\bar{x}-\bar{y}|) \exp\left(-\frac{|\bar{x}-\bar{y}|}{\xi}\right).$$

(A) "Tree-level", $O(\lambda^0)$

$\bar{x}$ —— $\bar{y}$

$$\langle \phi(\bar{x}) \phi(\bar{y}) \rangle_0 = \int_{p,q} e^{i\bar{p}\cdot\bar{x}} e^{i\bar{q}\cdot\bar{y}} \langle \phi(\bar{p}) \phi(\bar{q}) \rangle_0$$

$$\equiv \frac{1}{V^2} \sum_{p,q} e^{i\bar{p}\cdot\bar{x} + i\bar{q}\cdot\bar{y}} \, V \delta_{\bar{p}+\bar{q},0} \frac{1}{\tilde{p}^2 + m^2}$$

$$= \frac{1}{V} \sum_p e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{\tilde{p}^2 + m^2} \qquad ; \bar{p} = \frac{2\pi}{a} \cdot \frac{n}{N}$$

In order to evaluate the sum, let us

(a) take an infinite lattice, $N \to \infty$ p. 14 $\Rightarrow$ $\frac{1}{V}\sum_p \to \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d}$

$$\langle \phi(\bar{x}) \phi(\bar{y}) \rangle = \int_{-\pi/a}^{\pi/a} \frac{d^3 p}{(2\pi)^3} e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{\tilde{p}^2 + m^2}.$$

(b) consider large distances, $|\bar{x}-\bar{y}| \gg a$.
Then $\exp[i\bar{p}\cdot(\bar{x}-\bar{y})]$ oscillates very rapidly unless $|\bar{p}| \ll \frac{1}{a}$.
But for $|\bar{p}| \ll \frac{1}{a}$, $\tilde{p}_i = \frac{2}{a} \sin\frac{a p_i}{2} \sim p_i$!

Therefore the large-distance behaviour can be obtained from

$$\langle \phi(\bar{x})\phi(\bar{y})\rangle \approx \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3}\, e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{p^2+m^2} ,$$

when we have also extended the integration range by letting $a \to 0$.

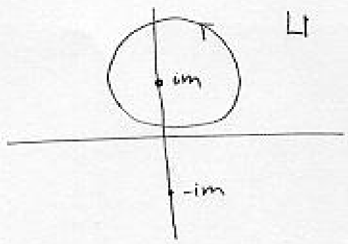
Go to radial coordinates: $\int d^3p = 2\pi \int_{-1}^{+1} d(\cos\theta) \int_0^{\infty} p^2\, dp$

$$\Rightarrow \frac{1}{(2\pi)^2} \int_{-1}^{+1} d(\cos\theta) \int_0^{\infty} dp\, p^2 \frac{1}{p^2+m^2}\, e^{ip|\bar{x}-\bar{y}|\cos\theta}$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} dp\, p^2 \frac{1}{p^2+m^2} \cdot \frac{1}{ip|\bar{x}-\bar{y}|} \left(e^{ip|\bar{x}-\bar{y}|} - e^{-ip|\bar{x}-\bar{y}|}\right)$$

$$= \frac{1}{(2\pi)^2} \frac{1}{i|\bar{x}-\bar{y}|} \cdot \int_{-\infty}^{\infty} dp\, p \frac{1}{p^2+m^2}\, e^{ip|\bar{x}-\bar{y}|}$$

Close the contour in the upper half plane



$$= \frac{1}{4\pi^{2}\cancel{i}} \frac{1}{|\bar{x}-\bar{y}|} \cdot 2\pi\cancel{i} \cdot \frac{\cancel{im}}{2\cancel{im}}\, e^{-m|\bar{x}-\bar{y}|}$$

$$= \frac{1}{4\pi|\bar{x}-\bar{y}|}\, e^{-m|\bar{x}-\bar{y}|}$$

$$\Rightarrow \quad \xi = \frac{1}{m}$$

(B) "1-loop level", $O(\lambda^1)$

$$S_I = \sum_{\bar z} a^d \, \frac{1}{4} \lambda \phi^4(\bar z) \quad , \quad \exp(-S_I) = 1 - S_I \, ,$$

$$\left. \frac{\langle \phi(\bar x)\phi(\bar y)\, e^{-S_I} \rangle_0}{\langle e^{-S_I} \rangle_0} \right|_{O(\lambda)} \overset{\text{p. 16}}{=} \langle S_I \rangle_0 \langle \phi(\bar x)\phi(\bar y) \rangle_0 - \langle \phi(\bar x)\phi(\bar y)\, S_I \rangle_0$$

It is useful to illustrate the computation with "Feynman graphs".

$$\langle S_I \rangle_0 \langle \phi(\bar x)\phi(\bar y) \rangle_0 = \langle \phi(\bar x)\phi(\bar y) \rangle_0 \cdot \frac{\lambda}{4} \cdot \int_{\bar z} \langle \underbrace{\phi(\bar z)\phi(\bar z)}_{3} \underbrace{\phi(\bar z)\phi(\bar z)} \rangle_0$$

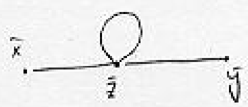
$\equiv$  $\frac{3}{4}\lambda \int_{\bar z}$  , $\int_{\bar z} = \sum_{\bar z} a^d$

$$- \langle \phi(\bar x)\phi(\bar y)\, S_I \rangle_0 = -\frac{\lambda}{4} \int_{\bar z} \langle \phi(\bar x)\phi(\bar y)\, \phi(\bar z)\phi(\bar z)\phi(\bar z)\phi(\bar z) \rangle_0$$

$$= -\frac{\lambda}{4} \int_{\bar z} \langle \overbrace{\phi(\bar x)\phi(\bar y)} \overbrace{\phi(\bar z)\phi(\bar z)}^{3} \overbrace{\phi(\bar z)\phi(\bar z)} \rangle_0$$

$$-\frac{\lambda}{4} \int_{\bar z} \langle \overbrace{\phi(\bar x)\phi(\bar y)\phi(\bar z)}^{4} \phi(\bar z)\, \overbrace{\phi(\bar z)\phi(\bar z)} \rangle_0 \quad (\text{contraction } \phi(\bar y)\text{–}\phi(\bar z): 3)$$

$\equiv -\frac{3}{4}\lambda \int_{\bar z}$  

$- 3\lambda \int_{\bar z}$ 

The first "disconnected" contribution cancels against $\langle S_I \rangle_0 \langle \phi(\bar x)\phi(\bar y) \rangle_0$!

Write the other one in momentum space:

$$-3\lambda \int_{\bar{z}} \int_{\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{p}_4,\bar{p}_5,\bar{p}_6} e^{i\bar{p}_1\cdot\bar{x}} e^{i\bar{p}_2\cdot\bar{y}} e^{i(\bar{p}_3+\bar{p}_4+\bar{p}_5+\bar{p}_6)\cdot\bar{z}} \langle\phi(\bar{p}_1)\phi(\bar{p}_3)\rangle_0 \langle\phi(\bar{p}_2)\phi(\bar{p}_4)\rangle_0 \langle\phi(\bar{p}_5)\phi(\bar{p}_6)\rangle_0$$

$$= -3\lambda \int_{\bar{p}_1,\bar{p}_2,\bar{p}_3,\bar{p}_4,\bar{p}_5,\bar{p}_6} e^{i\bar{p}_1\cdot\bar{x}} e^{i\bar{p}_2\cdot\bar{y}} \delta_{\bar{p}_3+\bar{p}_4+\bar{p}_5+\bar{p}_6}\, \delta_{\bar{p}_1+\bar{p}_3} \frac{1}{\tilde{p}_1^2+m^2}\, \delta_{\bar{p}_2+\bar{p}_4} \frac{1}{\tilde{p}_2^2+m^2}\, \delta_{\bar{p}_5+\bar{p}_6} \frac{1}{\tilde{p}_6^2+m^2}$$

$$\bar{p}_6 = -\bar{p}_5$$
$$\bar{p}_4 = -\bar{p}_2$$
$$\bar{p}_3 = -\bar{p}_1$$

$$= -3\lambda \int_{\bar{p}_1,\bar{p}_2,\bar{p}_5} e^{i\bar{p}_1\cdot\bar{x}} e^{i\bar{p}_2\cdot\bar{y}} \delta_{\bar{p}_1+\bar{p}_2} \frac{1}{\tilde{p}_1^2+m^2} \frac{1}{\tilde{p}_2^2+m^2} \frac{1}{\tilde{p}_5^2+m^2}$$

$$= -3\lambda \int_{\bar{p}} e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{(\tilde{p}^2+m^2)^2} \cdot \int_{\bar{q}} \frac{1}{\tilde{q}^2+m^2}$$

Let us now sum together with the tree-level term:

$$\langle\phi(\bar{x})\phi(\bar{y})\rangle = \int_{\bar{p}} e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \left[\frac{1}{\tilde{p}^2+m^2} - 3\lambda \frac{1}{(\tilde{p}^2+m^2)^2} \int_{\bar{q}} \frac{1}{\tilde{q}^2+m^2} \right] + O(\lambda^2)$$

$$= \int_{\bar{p}} e^{i\bar{p}\cdot(\bar{x}-\bar{y})} \frac{1}{\tilde{p}^2+m^2+3\lambda \int_{\bar{q}} \frac{1}{\tilde{q}^2+m^2}} + O(\lambda^2) \quad !$$

$$\Rightarrow \quad \xi = \frac{1}{\left(m^2+3\lambda \int_{\bar{q}} \frac{1}{\tilde{q}^2+m^2}\right)^{\frac{1}{2}}} + O(\lambda^2) \quad !$$

Let us finally look at the remaining integral, again for $N \to \infty$:

$$A(m^2) \equiv \int_{-\pi/a}^{\pi/a}\int_{-\pi/a}^{\pi/a}\int_{-\pi/a}^{\pi/a} \frac{d^3q}{(2\pi)^3} \; \frac{1}{\tilde{q}^2 + m^2} \quad , \quad \tilde{q}^2 = \sum_{i=1}^{3} \tilde{q}_i^2 , \quad \tilde{q}_i = \frac{2}{a}\sin\frac{a}{2}q_i .$$

This time we cannot take $a \to 0$, since there is no oscillating factor.

The integral can, however, be expanded in $\mathcal{O}(am)$!

By dimensional analysis,

$$A(m^2) = C_0 \cdot \frac{1}{a} + C_1 \cdot m + C_2 \cdot am^2 + \dots$$

$$C_0 = a \cdot \int_{-\pi/a}^{\pi/a}\int_{-\pi/a}^{\pi/a}\int_{-\pi/a}^{\pi/a} \frac{d^3q}{(2\pi)^3} \; \frac{1}{\frac{4}{a^2}\left(\sin^2\frac{a}{2}q_1 + \sin^2\frac{a}{2}q_2 + \sin^2\frac{a}{2}q_3\right)}$$

$$\overset{q = \frac{r}{a}}{=} \frac{1}{4} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{d^3r}{(2\pi)^3} \; \frac{1}{\sin^2\frac{r_1}{2} + \sin^2\frac{r_2}{2} + \sin^2\frac{r_3}{2}}$$

Exercise: express this integral in terms of a complete elliptic integral of the first kind —
or, evaluate it numerically:

$$C_0 = \frac{1}{4\pi} \cdot \Sigma , \qquad \Sigma = 3.17591153562 5 \dots$$

The coefficient C_1 can, miraculously, be correctly evaluated by letting $a \to 0$ and by simply ignoring the fact that the integral is divergent:

$$C_1 = \frac{1}{m} \cdot \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2+m^2} = \frac{1}{m} \cdot \frac{1}{8\pi^3} \cdot 4\pi \int_0^\infty dq \, \frac{q^2}{q^2+m^2}$$

$$= \frac{1}{m} \cdot \frac{1}{4\pi^2} \cdot \int_{-\infty}^{\infty} dq \, \frac{q^2}{q^2+m^2} = \frac{1}{m} \cdot \frac{1}{4\pi^2} \cdot 2\pi i \, \frac{(im)^2}{2im} = -\frac{1}{4\pi} .$$

Etc.