# Weak coupling expansion

Discretised field theories of the type introduced cannot, in general, be solved analytically. The solution can, however, be approximated in various ways. We start with analytic methods.

For $$S = \sum_{\bar{x}} a^d \left[ \frac{1}{2} \sum_{i=1}^{d} \frac{|\bar{\phi}(\bar{x}+a\hat{i}) - \bar{\phi}(\bar{x})|^2}{a^2} + \frac{1}{2} m^2 |\bar{\phi}(\bar{x})|^2 + \frac{1}{4} \lambda |\bar{\phi}(\bar{x})|^4 \right],$$

Weak coupling expansion is a solution in a power series in $\lambda$.

For simplicity of notation, consider a real scalar field, $\bar{\phi} \to \phi$.

Write $S = S_0 + S_I$, $\quad e^{-S_I} = 1 - S_I + \frac{1}{2} S_I^2 + \ldots$

Compute observables $\equiv$ expectation values $\equiv$ Green's functions:

$$\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n) \rangle = \frac{\int \{d\phi_{\bar{x}}\}\, \phi(z_1)\phi(z_2)\ldots\phi(z_n)\, e^{-S_0} e^{-S_I}}{\int \{d\phi_{\bar{x}}\}\, e^{-S_0} e^{-S_I}}$$

$$= \frac{\int \{d\phi_{\bar{x}}\} e^{-S_0} \left[ \phi(z_1)\phi(z_2)\ldots\phi(z_n) - \phi(z_1)\phi(z_2)\ldots\phi(z_n) S_I + \ldots \right]}{\int \{d\phi_{\bar{x}}\} e^{-S_0} \left[ 1 - S_I + \ldots \right]}$$

$$\equiv \frac{\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n) \rangle_0 - \langle \phi(z_1)\phi(z_2)\ldots\phi(z_n) S_I \rangle_0 + \ldots}{\langle 1 \rangle_0 - \langle S_I \rangle_0 + \ldots}$$

$$= \underbrace{\frac{\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n) \rangle_0}{\langle 1 \rangle_0}}_{O(\lambda^0)} \underbrace{- \frac{\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n) S_I \rangle_0}{\langle 1 \rangle_0} + \frac{\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n) \rangle_0}{\langle 1 \rangle_0} \frac{\langle S_I \rangle_0}{\langle 1 \rangle_0}}_{O(\lambda^1)} \underbrace{+ \ldots}_{O(\lambda^2)}$$

In order to evaluate such expectation values, we have to:

(1) Use Wick's theorem to relate all such expectation values to propagators, $G(z_1,z_2) \equiv \frac{\langle \phi(z_1)\phi(z_2)\rangle_0}{\langle 1 \rangle_0}$.

(2) Find an expression for the propagator in coordinate space (or configuration space, $z_1, z_2, \ldots$) in terms of a propagator in momentum space (or Fourier space, $k_1, k_2, \ldots$)

(3) Solve analytically for the momentum space propagator.

In this way get an expression for $\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n)\rangle$ in terms of momentum space propagators.

Notation: $x, y, z, \ldots$ = coordinate space

$k, p, q, \ldots$ = momentum space

## Wick's theorem:

How to write a general Green's function $\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n)\rangle_0$ in terms of 2-point correlators, or propagators, when the weight ($S_0$) is quadratic?

- With an odd number of "fields", $\phi(z_i)$, the result is clearly zero.
- The trick to be used: source terms!
- The result: $\langle \phi(z_1)\phi(z_2)\ldots\phi(z_n)\rangle_0 = \sum\limits_{\text{all combinations}} \langle \phi(z_1)\phi(z_2)\rangle_0 \langle \cdots \rangle_0$

Let us demonstrate this by collecting $\phi(\bar{x})$, $\forall \bar{x}$, into a single vector $v$.
Then $S_0 = \frac{1}{2} v^T A v$, where $A$ is a matrix. We assume $A^{-1}$ exists, and $A^T = A$.

$$\int dv\, e^{-\frac{1}{2}v^T A v + b^T v} \overset{v \to v + A^{-1}b}{=} \int dv\, e^{-\frac{1}{2}v^T A v - \cancel{\frac{1}{2}v^T b} - \cancel{\frac{1}{2}b^T v} - \frac{1}{2}b^T A^{-1} b + \cancel{b^T v} + b^T A^{-1} b}$$

$$= e^{+\frac{1}{2}b^T A^{-1} b} \int dv\, e^{-\frac{1}{2}v^T A v}$$

$$\Leftrightarrow \int dv\, e^{-\frac{1}{2}v_i A_{ij} v_j + b_i v_i} = e^{\frac{1}{2} b_i (A^{-1})_{ij} b_j} \int dv\, e^{-\frac{1}{2}v^T A v} \equiv e^{-W(b_i)}$$

$$\Rightarrow \langle v_i v_j \ldots v_k \rangle_0 = \frac{\int dv\,(v_i v_j \cdots v_k)\, e^{-\frac{1}{2}v_i A_{ij} v_j}}{\int dv\, e^{-\frac{1}{2}v_i A_{ij} v_j}} = \frac{\left(\frac{d}{db_i}\frac{d}{db_j}\cdots\frac{d}{db_k}\right) e^{-W(b_i)}\Big|_{b_i=0}}{e^{-W(0)}}$$

$$= \left(\frac{d}{db_i}\frac{d}{db_j}\cdots\frac{d}{db_k}\right) e^{\frac{1}{2}b_i (A^{-1})_{ij} b_j}\Big|_{b_i=0}$$

$$= \left(\frac{d}{db_i}\frac{d}{db_j}\cdots\frac{d}{db_k}\right)\left(1 + \frac{1}{2}b_i (A^{-1})_{ij} b_j + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 b_i (A^{-1})_{ij} b_j b_k (A^{-1})_{kl} b_l + \ldots\right)\Big|_{b_i=0}$$

$\Rightarrow$
- only one term in the series contributes
- the result is that all combinations arise. E.g.:

$$\frac{d}{db_i}\frac{d}{db_j}\frac{d}{db_k}\frac{d}{db_l}\;\frac{1}{8} b_i (A^{-1})_{ij} b_j b_k (A^{-1})_{kl} b_l$$

$$= \frac{d}{db_i}\frac{d}{db_j}\frac{d}{db_k}\;\frac{1}{2}(A^{-1})_{lj} b_j b_k (A^{-1})_{kl} b_l$$

$$= \frac{d}{db_i}\frac{d}{db_j}\left[\frac{1}{2}(A^{-1})_{lk} b_k (A^{-1})_{kl} b_l + (A^{-1})_{lj} b_j (A^{-1})_{kl} b_l\right]$$

$$= \frac{d}{db_i}\left[(A^{-1})_{lk}(A^{-1})_{jl} b_l + (A^{-1})_{lj}(A^{-1})_{kl} b_l + (A^{-1})_{lj} b_j (A^{-1})_{ki}\right]$$

$$= (A^{-1})_{lk}(A^{-1})_{ji} + (A^{-1})_{lj}(A^{-1})_{ki} + (A^{-1})_{li}(A^{-1})_{kj}$$

$$= v_i v_j v_k v_l + v_i v_j v_k v_l + v_i v_j v_k v_l$$

Here $\underbrace{v_i\, v_j}_{} \equiv \langle v_i\, v_j \rangle_0 / \langle 1 \rangle_0 = (A^{-1})_{ji}$ .

Let us be more explicit about what this means:

- Coordinate space propagator in terms of momentum space one:

$$\langle \phi(z_1)\phi(z_2) \rangle_0 = \int_{p_1}\int_{p_2} e^{ip_1\cdot z_1} e^{ip_2\cdot z_2} \langle \phi(p_1)\phi(p_2) \rangle_0 .$$

- $S_0$ in momentum space:

$$S_0 = \sum_{\bar{x}} a^d \left[ \frac{1}{2}\cdot\frac{1}{a^2}\cdot \sum_{i=1}^{d} \frac{\left(\phi(\bar{x}+a\hat{i}) - \phi(\bar{x})\right)^2 + \left(\phi(\bar{x}) - \phi(\bar{x}-a\hat{i})\right)^2}{2} + \frac{1}{2} m^2 \phi^2(\bar{x}) \right]$$

we have made this more symmetric by changing summation indices.

$$= \sum_{\bar{x}} a^d \int_p\int_q \left\{ \frac{1}{4a^2} \sum_{i=1}^{d} \phi(p)\phi(q) e^{ip\cdot\bar{x}} e^{iq\cdot\bar{x}} \left[ e^{ip_i a} e^{iq_i a} - e^{ip_i a} - e^{iq_i a} + 2 - e^{-ip_i a} - e^{-iq_i a} + e^{-ip_i a} e^{-iq_i a} \right] + \frac{1}{2} m^2 \phi(p)\phi(q) e^{ip\cdot\bar{x}} e^{iq\cdot\bar{x}} \right\}$$

we have written all the terms in the momentum space

$\sum_{\bar{x}} a^d e^{i(p+q)\cdot\bar{x}} = V \delta_{p+q,0 \bmod \frac{2\pi}{a}}$ ; $V = L_1 L_2 \ldots L_d = (aN_1)(aN_2)\ldots(aN_d)$

$$= \int_{p,q} V \delta_{p+q,0 \bmod \frac{2\pi}{a}} \frac{1}{2}\phi(p)\phi(q) \left[ \frac{1}{2a^2} \sum_{i=1}^{d} \left( 4 - 2e^{ip_i a} - 2e^{-ip_i a} \right) + m^2 \right]$$

used $q = -p \bmod \frac{2\pi}{a}$

$$= \int_{p,q} V \delta_{p+q,0 \bmod \frac{2\pi}{a}} \frac{1}{2}\phi(p)\phi(q) \left[ \sum_{i=1}^{d} \left( \frac{e^{\frac{ip_i a}{2}} - e^{-\frac{ip_i a}{2}}}{a\, i} \right)^2 + m^2 \right]$$

$$= \int_{p,q} V \delta_{p+q,0 \bmod \frac{2\pi}{a}} \frac{1}{2}\phi(p)\phi(q) \left[ \sum_{i=1}^{d} \tilde{p}_i^{\,2} + m^2 \right] ;$$

$$\tilde{p}_i \equiv \frac{2}{a} \sin\left(\frac{a p_i}{2}\right) .$$

$\phi(x)$ is real $\Rightarrow$ $\phi^*(x) = \int_k e^{-ik\cdot x} \phi^*(k) \overset{k\to -k}{=} \int_k e^{ik\cdot x} \phi^*(-k) = \phi(x)$

$\Rightarrow \quad \phi^*(-k) = \phi(k)$ , or $\phi(-k) = \phi^*(k)$.

- $\int_q = \frac{1}{V} \sum_{q_n}$

$$\Rightarrow \quad S_0 = \int_p \frac{1}{2} |\phi(p)|^2 \left[ \sum_{i=1}^{d} \tilde{p}_i^2 + m^2 \right]$$
$$= \frac{1}{V} \sum_{p_n} \frac{1}{2} |\phi(p_n)|^2 \left[ \sum_{i=1}^{d} \tilde{p}_i^2 + m^2 \right]$$

- Since $\phi(p_n) = a^d \sum_x \phi(x) e^{-ip_n x}$ is a unitary transformation, we can choose $\phi(p_n)$ as integration variables. One should actually be a bit careful because $\phi(p_n)$ are complex and have thus two real components, while on the other hand $\phi(-p_n) = \phi^*(p_n)$ removes half of the $p_n$-modes. But the bottom line is that one can proceed as if $|\phi(p_n)|$ were real:

$$\frac{\langle \phi(p_1)\, \phi(p_2) \rangle_0}{\langle 1 \rangle_0} = \delta_{p_1+p_2,0} \frac{\langle |\phi(p_1)|^2 \rangle_0}{\langle 1 \rangle_0} = \delta_{p_1+p_2,0} \cdot (A^{-1})_{p_1 p_1} ,$$

where
$$S_0 = \frac{1}{2} v^T A v = \frac{1}{2} \begin{pmatrix} \cdots & |\phi(p_n)| & \cdots \end{pmatrix} \begin{pmatrix} \times & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & \times & 0 & 0 \\ 0 & 0 & & \times & 0 \\ 0 & 0 & 0 & * & 0 \end{pmatrix} \begin{pmatrix} \vdots \\ |\phi(p_n)| \\ \vdots \\ \vdots \end{pmatrix} ,$$

$$* = \frac{1}{V} \left[ \sum_{i=1}^{d} \tilde{p}_i^2 + m^2 \right] , \quad *^{-1} = \frac{V}{\left[ \sum_{i=1}^{d} \tilde{p}_i^2 + m^2 \right]}$$

$$\Rightarrow \boxed{\frac{\langle \phi(p_1)\, \phi(p_2) \rangle_0}{\langle 1 \rangle_0} = V \delta_{p_1+p_2,0} \cdot \frac{1}{\sum_{i=1}^{d} \tilde{p}_i^2 + m^2}}$$