Discretised field theory

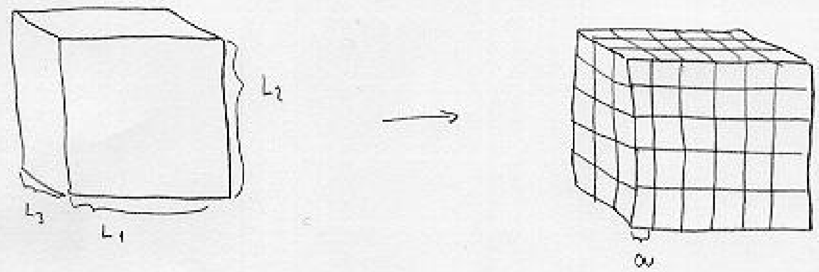
Let us now consider a different type of an action S:

$$S = \int_0^{L_1} dx_1 \int_0^{L_2} dx_2 \dots \int_0^{L_d} dx_d \left[\sum_{i=1}^{d} \frac{1}{2}\, \partial_i \bar{\phi} \cdot \partial_i \bar{\phi} + \frac{1}{2} m^2 \bar{\phi}\cdot\bar{\phi} + \frac{1}{4}\lambda \left(\bar{\phi}\cdot\bar{\phi}\right)^2 \right],$$

where $\bar{\phi}$ is a vector $\in \mathbb{R}^N$, $\partial_i \equiv \frac{\partial}{\partial x_i}$, and m^2, λ are parameters.

(Some physics motivation for this kind of an action will come from Quantum Field Theory.)

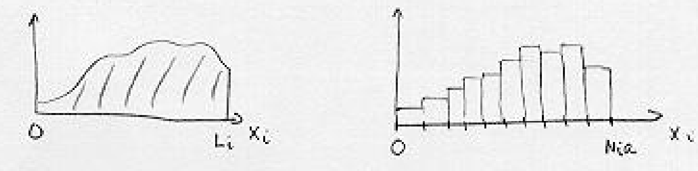
What happens when we "discretise", or "latticize", the box?



We call "a" the "lattice spacing"; $L_i = N_i \cdot a$, $N_i \in \mathbb{N}$.

If the lattice spacing a is small, we can work to first order in it:

- $$\int_0^{L_i} dx_i \; f(x_i) \approx \sum_{n_i=0}^{N_i-1} a\, f(a \cdot n_i)$$



- $$\partial_i f(x_i) = \frac{\partial}{\partial x_i} f(x_i) \approx \frac{f(x_i + a) - f(x_i)}{a}$$

Then the action becomes

$$S \approx \left\{ \prod_{k=1}^{d} \sum_{n_k=0}^{N_k-1} \right\} \cdot a^d \cdot \left[\frac{1}{2a^2} \sum_{i=1}^{d} \left(\bar{\phi}(\bar{x}+a\hat{i}) \cdot \bar{\phi}(\bar{x}+a\hat{i}) + \bar{\phi}(\bar{x}) \cdot \bar{\phi}(\bar{x}) - 2\bar{\phi}(\bar{x}) \cdot \bar{\phi}(\bar{x}+a\hat{i}) \right) + \frac{1}{2} m^2 \bar{\phi}(\bar{x}) \cdot \bar{\phi}(\bar{x}) + \frac{1}{4} \lambda \left(\bar{\phi}(\bar{x}) \cdot \bar{\phi}(\bar{x}) \right)^2 \right]$$

Let us also redefine the field variables:

$$\bar{\phi}(\bar{x}) \equiv \sqrt{2\varkappa a^{2-d}} \; \bar{s}(\bar{x})$$

And we note, denoting $\left\{ \prod_{k=1}^{d} \sum_{n_k=0}^{N_k-1} \right\} \equiv \sum_{\bar{x}}$ and assuming periodic boundary conditions, that

$$\sum_{\bar{x}} \bar{\phi}(\bar{x}+a\hat{i}) \cdot \bar{\phi}(\bar{x}+a\hat{i}) = \sum_{\bar{x}} \bar{\phi}(\bar{x}) \cdot \bar{\phi}(\bar{x})$$

Then

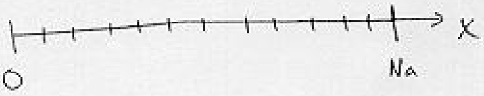
$$S = \sum_{\bar{x}} \left\{ -2\varkappa \sum_{i=1}^{d} \bar{s}(\bar{x}) \cdot \bar{s}(\bar{x}+a\hat{i}) + (2d + m^2a^2)\varkappa \, \bar{s}(\bar{x}) \cdot \bar{s}(\bar{x}) + \lambda a^{4-d} \varkappa^2 \left(\bar{s}(\bar{x}) \cdot \bar{s}(\bar{x}) \right)^2 \right\}$$

We observe that if we choose $2\varkappa \equiv \beta$ and restrict $\bar{s}(\bar{x}) \cdot \bar{s}(\bar{x}) = 1$, this is (apart from an inessential overall constant) just an $O(N)$ spin model with $\bar{h} = 0$!

In the following, we will consider this generalised "field theory" case where $\bar{s}(\bar{x})$ (or $\bar{\phi}(\bar{x})$) has arbitrary length.

Fourier analysis on the lattice

To study the properties of $S[\bar{\phi}]$, we need to work out Fourier-analysis in discretised space. We do this for $d=1$; generalisation will be obvious.



Periodic boundary conditions $\Rightarrow$ $Na \hat{=} 0$ $\Rightarrow$ independent sites are $x = (0, \dots, N-1)\cdot a$.

What kind of Fourier-modes are there? Look at e^{ikx}.

(a) periodicity: $\quad e^{ik\,Na} \equiv e^{ik\cdot 0} = 1 \quad \Rightarrow \quad k = \frac{2\pi}{Na}\cdot n, \quad n \in \mathbb{Z}$

(b) discreteness: $\quad$ since $x = m \cdot a$,

$$e^{ik\cdot x} = e^{i\frac{2\pi}{N}nm}$$

So for $n \to n+N$, the exponent is the same, and we do not obtain any new information

$\Rightarrow$ it is enough to consider the first "Brillouin Zone", $n = 0, \dots, N-1$.

Thus, we can write

$$\phi(x) \equiv \underbrace{\frac{1}{Na}}_{\text{by convention}} \sum_{n=0}^{N-1} \phi_n \, e^{i\frac{2\pi}{a}\cdot\frac{n}{N}\cdot x} \quad , \quad x = (0, \dots, N-1)a$$

Delta-function:

$$a \sum_{\frac{x}{a}=0}^{N-1} e^{i\frac{2\pi}{a}\cdot\frac{n}{N}\cdot x} = aN \; \delta_{n,\,0 \bmod N} \qquad \left(\text{use } \sum_{i=0}^{N-1} a^i = \frac{1-a^N}{1-a}\right)$$

Inverse transformation:

$$\phi_n = a \sum_{\frac{x}{a}=0}^{N-1} \phi(x)\, e^{-i\cdot\frac{2\pi}{a}\cdot\frac{n}{N}\cdot x}$$

Special limits:

(1) Infinite lattice: $N \to \infty$, a fixed, $L = Na \to \infty$:

denote $k = \frac{2\pi}{a} \cdot \frac{n}{N}$. Then $\sum_{n=0}^{N-1} \frac{1}{Na} f(k) = \frac{1}{2\pi} \cdot \sum_{n=0}^{N-1} \frac{2\pi}{a} \cdot \frac{1}{N} f(k)$

$$= \frac{1}{2\pi} \sum_{n=0}^{N-1} \Delta k \, f(k) \xrightarrow{N \to \infty} \frac{1}{2\pi} \int_0^{2\pi/a} dk \, f(k).$$

$$\Rightarrow \quad \cdot \ \phi(x) = \int_0^{\frac{2\pi}{a}} \frac{dk}{2\pi} \bar{\phi}(k) \, e^{ikx}$$

$$\cdot \ a \sum_{\frac{x}{a}=0}^{\infty} e^{ik \cdot x} = \lim_{N \to \infty} aN \, \delta_{\frac{aNk}{2\pi}, \, 0 \bmod N} \qquad (\text{Kronecker})$$

$$= \lim_{N \to \infty} aN \, \delta\left(\frac{aN}{2\pi}\left(k \bmod \frac{2\pi}{a}\right)\right) \qquad (\text{Dirac})$$

$$= 2\pi \, \delta\left(k \bmod \frac{2\pi}{a}\right) \qquad \left(\delta(ax) = \frac{1}{|a|}\delta(x)\right)$$

$$\cdot \ \bar{\phi}(k) = a \sum_x \phi(x) \, e^{-ikx}$$

(2) Continuum: $a \to 0$, $N \to \infty$, $aN = L$ fixed:

denote $k_n = \frac{2\pi}{aN} \cdot n = \frac{2\pi}{L} \cdot n$. Then $\sum_{n=0}^{N-1} \frac{1}{Na} f(k) \to \frac{1}{L} \sum_{n=0}^{\infty} f(k_n)$.

At the same time,

$$a \sum_{\frac{x}{a}=0}^{N-1} e^{ik_n x} = aN \, \delta_{n, \, 0 \bmod N}$$

$$\to \int_0^L dx \, e^{ik_n \cdot x} = L \, \delta_{k_n, 0}$$

$$\Rightarrow \quad \cdot \ \phi(x) = \frac{1}{L} \sum_{n=0}^{\infty} \bar{\phi}(k_n) \, e^{ik_n x}$$

$$\cdot \ \bar{\phi}(k_n) = \int_0^L dx \, e^{-ik_n x} \phi(x).$$

(3) Infinite volume <u>and</u> continuum :

- $\phi(x) = \int_0^\infty \frac{dk}{2\pi} \bar{\phi}(k) e^{ikx}$
- $\int_0^\infty dx \, e^{ik \cdot x} = 2\pi \delta(k)$
- $\bar{\phi}(k) = \int_0^\infty dx \, \phi(x) e^{-ikx}$

Notes: (a) In infinite volume and continuum, it is usually more convenient to be "symmetric", and define

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \bar{\phi}(k) e^{ik \cdot x}$$

$$\bar{\phi}(k) = \int_{-\infty}^{\infty} dx \, \phi(x) e^{-ikx}$$

(b) In general, the Brillouin Zone can also be chosen symmetric:

$$\phi(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dk}{2\pi} \bar{\phi}(k) e^{ik \cdot x}$$

Notation & generalisations

In the following, we will often write simply

$$\phi(x) = \int_k \phi(k) e^{ik \cdot x}$$

$$\phi(k) = \int_x \phi(x) e^{-ik \cdot x} ,$$

which apply in d-dimensional space as well, and assume the measure relevant for each context.

E.g.

$$\phi(x) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \cdots \frac{d^d k}{(2\pi)^d} \phi(\bar{k}) e^{i\bar{k} \cdot \bar{x}} .$$

From the arguments, k, p, q vs x, y, z, we remember whether it is a coordinate space function or its Fourier transform.