Global symmetries

Let us call minus the object appearing in the exponential, the action S:

$$S = -\beta \sum_{\bar{x}\in \text{lattice}} \sum_{i=1}^{d} \bar{s}_{\bar{x}} \cdot \bar{s}_{\bar{x}+\hat{i}} - \bar{h}\cdot \sum_{\bar{x}\in\text{lattice}} \bar{s}_{\bar{x}} ,$$

where $\hat{i}$ is unit vector in direction i and we have written out $\sum_{\langle ij\rangle}$ explicitly.

The integration / summation measure is denoted by

$$\sum_{\{\bar{s}_{\bar{x}}\}} \to \int \{d\bar{s}_{\bar{x}}\} \equiv \prod_{k=1}^{N_{tot}} \int_{S^{N-1}} d\bar{s}_k \qquad \text{for } O(N)$$

$$\prod_{k=1}^{N_{tot}} \sum_{\sigma_k=1}^{q} \qquad \text{for Potts}$$, similarly for Ising.

Then

$$Z = \int \{d\bar{s}_{\bar{x}}\} \, e^{-S} .$$

The system is said to have a (global) symmetry, if one can make a (linear) redefinition of the variables, $\bar{s}_{\bar{x}} \Rightarrow \bar{s}'_{\bar{x}} \equiv M\bar{s}_{\bar{x}}$, where M is a matrix, such that

(a) S is invariant
(b) $\{d\bar{s}_{\bar{x}}\}$ is invariant.

Example: "$O(N)$"-model :

* $\bar{s}_{\bar{x}} \cdot \bar{s}_{\bar{x}+\hat{i}} \longrightarrow \sum_i (\bar{s}'_{\bar{x}})_i (\bar{s}'_{\bar{x}+\hat{i}})_i = \sum_{i,j,k=1}^{N} M_{ij} M_{ik} (\bar{s}_{\bar{x}})_j (\bar{s}_{\bar{x}+\hat{i}})_k$

$$= \sum_{j,k=1}^{N} (\bar{s}_{\bar{x}})_j \, [M^T M]_{jk} \, (\bar{s}_{\bar{x}+\hat{i}})_k$$

The product is invariant if $[M^T M]_{jk} = \delta_{jk}$. Such matrices are called orthogonal $N\times N$-matrices, $M \in O(N)$.

* $\bar{h}\cdot\bar{s}_{\bar{x}} \rightarrow \sum_{j,k=1}^{N} h_j M_{jk} (\bar{s}_{\bar{x}})_k \Rightarrow$ not invariant !

* $\int d\bar{s}_{\bar{x}} = \int_{|\bar{s}_{\bar{x}}|=1} d^N s_{\bar{x}} \rightarrow \int_{|\bar{s}'_{\bar{x}}|=1} d^N s'_{\bar{x}} = |\det M| \int_{|\bar{s}_{\bar{x}}|=1} d^N s_{\bar{x}}$

If $M^T M = \mathbb{1}$, then $\det M = \pm 1$!

Therefore, $O(N)$-model has a global $O(N)$-symmetry, if $\bar{h} = \bar{0}$!

Symmetry breaking

What is $\langle \vec{M} \rangle = \langle \sum_{\bar{x}} \bar{s}_{\bar{x}} \rangle$?

* Suppose that $\bar{h} = 0$. Then the system does have a symmetry, and

$$\Rightarrow \langle \vec{M} \rangle = \frac{\int \{d\bar{s}'_{\bar{x}}\} \sum_{\bar{y}} \bar{s}'_{\bar{y}} \, e^{-S[\bar{s}'_{\bar{x}}]}}{\int \{d\bar{s}'_{\bar{x}}\} e^{-S[\bar{s}'_{\bar{x}}]}} = M \cdot \langle \vec{M} \rangle .$$

But the matrix $\mathrm{diag}_{N\times N}(-1,-1,\dots,-1) \in O(N) \Rightarrow \langle \vec{M} \rangle = -\langle \vec{M} \rangle \Rightarrow \langle \vec{M} \rangle = 0$.
So symmetry is restored. The system is in a disordered phase.

* Suppose that $\bar{h} \neq 0$. Then the symmetry is broken explicitly. As long as $N_{tot} < \infty$, the partition function is an analytic function of its arguments. It is also even in $\bar{h}$, so that

$$\langle \vec{M} \rangle = c_1(\beta, N_{tot})\, \bar{h} + c_2(\beta, N_{tot})\, h^2 \bar{h} + \dots$$

* What happen if $N_{tot} \to \infty$?

The symmetry is said to be broken spontaneously, if

$$\lim_{\bar{h} \to 0} \left\{ \lim_{N_{tot} \to \infty} \left[\frac{\langle \vec{M} \rangle}{N_{tot}} \cdot \frac{\bar{h}}{|\bar{h}|} \right] \right\} \neq 0 .$$

Example: in the $d=1$ Ising model (p.5), the symmetry is "almost" spontaneously broken, but not really:

$$\lim_{N_{tot} \to \infty} \left[\frac{\langle \vec{M} \rangle}{N_{tot}} \cdot \frac{\bar{h}}{|\bar{h}|} \right] = \frac{\sinh(|\bar{h}|)}{\sqrt{\sinh^2(|\bar{h}|) + e^{-4\beta}}} .$$

Nature of a phase transition

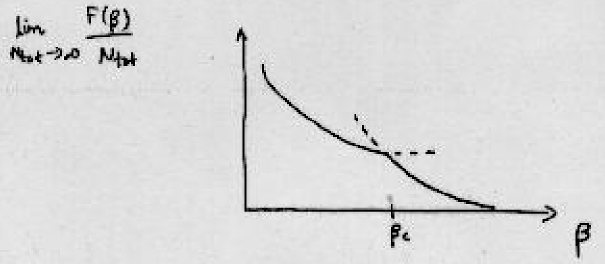
Whether the symmetry is broken or not, depends often on the value of β. In this case we say that the system has a phase transition at some $\beta \equiv \beta_c$.

Phase transitions can be of different types:

- Discontinuous, of first order phase transitions

  * physically: two "regular" but different phases coexist at β_c; say, ice and water.

  * mathematically: while $Z(\beta)$, or $F(\beta)$, is continuous, its derivatives are not.



- Continuous, or second order phase transitions

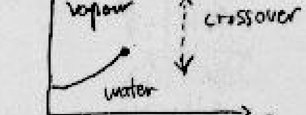
  * physically: there is one phase but it goes through a singularity at β_c; say, correlation lengths ξ diverge, and there are large fluctuations on all length scales at β_c.

  * mathematically: while $F'(\beta)$ is continuous, $F^{(2)}(\beta)$ is not.

  [One could similarly define 3rd order, ..., phase transitions, but the academic nature of this definition is reflected by the fact that such transitions are usually not observed in nature.]

- Crossover

  * no phase transition, just a rapid change in the properties of the system. Say, water $\to$ vapour at pressures beyond the critical point.



Characteristics in terms of basic observables

· Discontinuous

* $\lim\limits_{\beta \to \beta_c^-} \dfrac{\langle E \rangle}{N_{tot}} - \lim\limits_{\beta \to \beta_c^+} \dfrac{\langle E \rangle}{N_{tot}} \neq 0$. This difference is called the latent heat

* $\lim\limits_{\beta \to \beta_c^-} \dfrac{\langle M \rangle}{N_{tot}} = 0 \ , \quad \lim\limits_{\beta \to \beta_c^+} \dfrac{\langle M \rangle}{N_{tot}} \neq 0 .$

* $\chi_E = \dfrac{1}{N_{tot}} \left\langle (E - \langle E \rangle)^2 \right\rangle \approx N_{tot} \left[\lim\limits_{\beta \to \beta_c^-} \dfrac{\langle E \rangle}{N_{tot}} - \lim\limits_{\beta \to \beta_c^+} \dfrac{\langle E \rangle}{N_{tot}} \right]^2 \propto N_{tot} .$

* $\chi_M = \dfrac{1}{N_{tot}} \left\langle (M - \langle M \rangle)^2 \right\rangle \approx N_{tot} \cdot \left[\dfrac{\langle M \rangle}{N_{tot}} \right]^2 \propto N_{tot} .$

(The latter two follow because there are always fluctuations where the system is in the "other" phase.)

* $\lim\limits_{\beta \to \beta_c^-} \xi - \lim\limits_{\beta \to \beta_c^+} \xi \neq 0$

· Continuous

* $\lim\limits_{\beta \to \beta_c^-} \dfrac{\langle E \rangle}{N_{tot}} = \lim\limits_{\beta \to \beta_c^+} \dfrac{\langle E \rangle}{N_{tot}} .$

* $\dfrac{\langle M \rangle}{N_{tot}} = 0$ for $\beta < \beta_c$, $\quad \dfrac{\langle M \rangle}{N_{tot}} \approx A |\tau|^{\delta}$ for $\beta > \beta_c$, where $\tau \equiv \beta_c^{-1} - \beta^{-1}$.

* $\chi_E = B \, |\tau|^{-\alpha} + \ldots$

* $\chi_M = C \, |\tau|^{-\gamma} + \ldots$

* $\xi = D \cdot |\tau|^{-\nu} + \ldots$

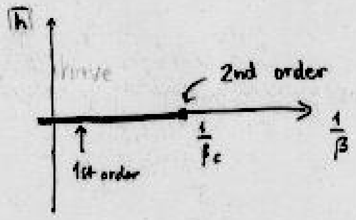
The numbers $\alpha, \gamma, \nu, \delta$ are called critical exponents.

Examples

- $d=1$ Ising-model does not have a phase transition (see p.5).
- $d=2$ Ising has a 2nd order transition at $\beta_c = \frac{1}{2}\ln(1+\sqrt{2})$.
  The symmetry which breaks is a discrete one: Ising $= O(1) = \mathbb{Z}(2) = \{-1,+1\}$
- $d=2$ $O(N)$ models, $N \geq 2$, do not have a real symmetry breaking phase transition. This is due to the Mermin-Wagner-Coleman theorem: Continuous symmetries do not break in two dimensions.

  There is however still something of a phase transition: it is called the Kosterlitz-Thouless transition, a special case.

- $d=3$ $O(N)$ models have all second order phase transitions. The corresponding critical exponents have been determined numerically, and with various approximative analytic procedures.



If one rather fixes β and changes $\bar{h}$, one can find 1st order transitions.

- $d=2$ Potts models have 2nd order transitions for $q \leq 4$, first order for $q \geq 5$. $d=3$ Potts 2nd order for $q=2$, first for $q \geq 3$.

- $d \geq 4$ $O(N)$ models have second order phase transitions, with "mean field" (analytically computable) exponents.

Universality: All models with the same symmetry and d, have phase transitions with equivalent characteristics.

Summary: The simple $d=3$ $O(3)$ model does describe the qualitative features of the 2nd order phase transition of ferromagnets at the Curie point.