Lattice Field Theory

Mikko Laine (University of Bielefeld, Germany)

1. Why and how? Scalar $\lambda\phi^4$ on the lattice.
2. The perfect field theory: discretised pure SU(N_c).
3. What should we like to do for the real world?
4. Theoretical challenge: chiral symmetry.
5. Practical challenge: hadron spectra from QCD.

Lecture 1

Why and how?
Scalar $\lambda\phi^4$ on the lattice.

Q: Why do we need another regularization scheme?

Theoretician: Lattice regularization can provide a constructive non-perturbative “no-magic” **definition** of what we mean by quantum field theory.

This allows (in principle) to pose conceptually sound and mathematically precise **questions**.

Phenomenologist: Since the questions are well-posed, unambiguous **answers** exist, and the lattice framework offers a constructive method of finding them out.

Theoretician: We could even do this analytically.

P: And if not clever enough, then the lattice also allows (in many cases) to obtain an approximate **numerical** answer, with an (in principle) reliable error bar.

T: Doing numerics is not the main goal, though!

As a simple example of how it goes, consider a transition amplitude in scalar field theory.[1]

$$
\begin{aligned}
\langle\phi_b|e^{-i\hat{H}T}|\phi_a\rangle &= \int_{\phi(0,\mathbf{x})=\phi_a}^{\phi(T,\mathbf{x})=\phi_b} \mathcal{D}\phi\, e^{i\int_0^T \mathrm{d}t \int_V \mathrm{d}^3\mathbf{x}\, \mathcal{L}_M} , \\
\mathcal{L}_M &= \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) .
\end{aligned}
$$

Let us try to deform this in a way that the integral becomes manifestly finite; i.e., let us try to give a precise meaning to this formal expression.

[1]The derivation of the path integral already involves a discretised time direction; here we go in the "opposite" direction, and discretise space too.

First: analytic continuation (Wick rotation).

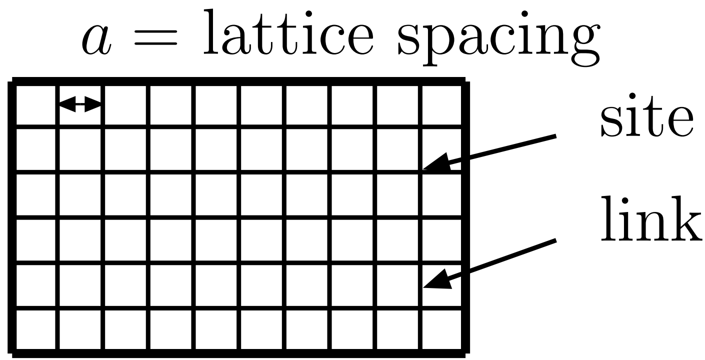
$$t \equiv -i\tau \ , \quad \tau \in \mathbb{R} \ ; \quad i\mathrm{d}t = \mathrm{d}\tau \ , \quad T \equiv -i\beta \ ,$$

$$\langle \phi_b | e^{-\beta \hat{H}} | \phi_a \rangle = \int_{\phi(0,\mathbf{x})=\phi_a}^{\phi(\beta,\mathbf{x})=\phi_b} \mathcal{D}\phi \, e^{-\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x} \, \mathcal{L}_E} \ ,$$

$$\begin{aligned} \mathcal{L}_E &= -\mathcal{L}_M(it \to \tau) \\ &= \frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \\ &\equiv \frac{1}{2}\partial_\mu \phi \, \partial_\mu \phi + V(\phi) \ . \end{aligned}$$

Now $\mathcal{L}_E \geq 0 \Rightarrow$ fluctuations exponentially suppressed.

Second: introduce lattice.



Fields live either on sites or on links.

Discretise derivatives:

$$\partial_\mu \phi \to \Delta_\mu \phi \equiv \frac{1}{a}[\phi(x + a\hat{\mu}) - \phi(x)] \, .$$

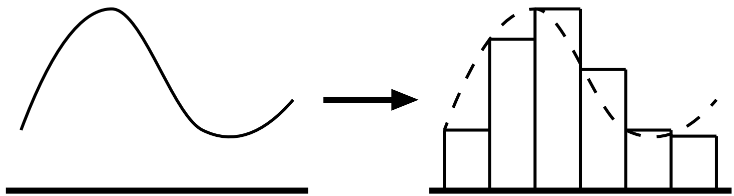
Of course there are other possibilities as well, like

$$\partial_\mu \phi \to \Delta^*_\mu \phi \equiv \frac{1}{a}[\phi(x) - \phi(x - a\hat{\mu})] \, .$$

Both discretisations have errors of $\mathcal{O}(a)$. Errors can be reduced arbitrarily, but then it is not enough to connect **nearest neighbours**.

Discretise integration:

$$\int \mathrm{d}x\, \mathcal{L}_E(\phi(x)) \approx a \sum_{x_i} \mathcal{L}_E(\phi(x_i)) \; .$$



This recipe is again not unique, but ambiguities disappear for $a \to 0$.

The resulting multidimensional integral:

$$\langle \phi_b | e^{-\beta \hat{H}} | \phi_a \rangle \approx \int_{\phi(0,\mathbf{x})=\phi_a}^{\phi(\beta,\mathbf{x})=\phi_b} \mathcal{D}\phi \, e^{-S_E} \ ,$$

$$S_E = a^4 \sum_x \left\{ \frac{1}{2} \sum_\mu [\Delta_\mu \phi(x)]^2 + V(\phi(x)) \right\}$$

$$= \sum_{x,\mu} \frac{a^2}{2} \left[\phi^2(x + a\hat{\mu}) + \phi^2(x) - 2\phi(x)\phi(x + a\hat{\mu}) \right]$$

$$+ \sum_x a^4 V(\phi(x)) \ .$$

What has happened? Let us look at perturbation theory. Fields defined only at discrete points:

$$\begin{aligned}\phi(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \tilde{\phi}(p)\, e^{ip\cdot x} \;, \\ \tilde{\phi}(p) &= a^4 \sum_x \phi(x)\, e^{-ip\cdot x} \;, \\ x &= a(n_0, n_1, n_2, n_3) \;, \quad n_\mu \in \mathbb{Z} \;.\end{aligned}$$

We see that $p_\mu \to p_\mu + 2\pi/a$ does not bring in anything new (Brillouin zone).

$$\Rightarrow \; \phi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \tilde{\phi}(p)\, e^{ip\cdot x} \;.$$

Next, find the propagator. For $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$, the quadratic part of the action becomes [exercise]

$$S_E^{(2)} = \frac{1}{2}\int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\tilde{\phi}(-p)[\tilde{p}^2 + m^2]\tilde{\phi}(p)\ ,$$

$$\tilde{p}^2 = \sum_\mu \tilde{p}_\mu^2\ , \qquad \tilde{p}_\mu = \frac{2}{a}\sin\frac{ap_\mu}{2}\ .$$

For $p_\mu \to 0$, $\tilde{p}_\mu \approx p_\mu$; for $p_\mu \to \pm\frac{\pi}{a}$, $\tilde{p}_\mu \approx \pm\frac{2}{a}$.

Propagator:

$$\frac{1}{\tilde{p}^2 + m^2} \, .$$

A typical loop integral:

$$\int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\tilde{p}^2 + m^2} \, .$$

So everything is manifestly finite; moreover

$$0 \leq \tilde{p}^2 \leq 4\left(\frac{2}{a}\right)^2 \, .$$

Momentum cutoff is set by inverse lattice spacing.

Note: finite lattice spacing $a \equiv$ UV regularization.

Oftentimes one also introduces a **finite volume**.
But this is no UV-regularization – this is IR physics!

Related to volume are boundary conditions. A common choice is to sum over all possibilities with **periodic boundary conditions**, which yields nothing but the finite-temperature partition function:

$$\int_{-\infty}^{\infty} \mathrm{d}\phi \, \langle \phi | e^{-\beta \hat{H}} | \phi \rangle = \mathrm{Tr}[e^{-\beta \hat{H}}] \equiv \mathcal{Z}(\beta, V) \; .$$

Everything looks fine now, except that it's not unique.
We need to take $a \to 0$ to remove lattice artifacts and reach the **universal** continuum limit.

The question of **renormalizability** becomes:
Does the continuum limit exist?
Is it non-trivial?

Let us rename our objects to be "bare", ϕ_0, m_0^2, λ_0:

$$S_E = a^4 \sum_x \left\{ \frac{1}{2} \sum_\mu (\Delta_\mu \phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{4} \lambda_0 \phi_0^4 \right\} .$$

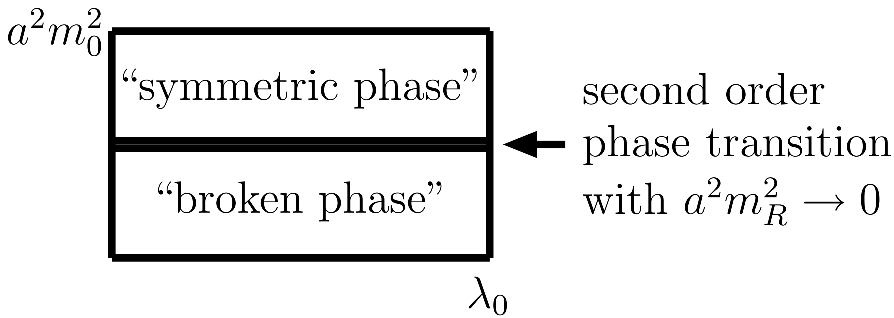
After rescaling the integration variable as $\hat{\phi}_0 \equiv a\phi_0$, the outcome depends only on $(am_0)^2$ and λ_0.

We can also define **renormalised parameters**, e.g.:

$$a^4 \sum_x \langle \phi_0(x) \phi_0(0) \rangle e^{-ip\cdot x} \equiv \frac{Z_R}{m_R^2 + p^2 + \mathcal{O}(p^4)} ,$$

$$a^{12} \sum_{x,y,z} \langle \phi_0(x) \phi_0(y) \phi_0(z) \phi_0(0) \rangle e^{-i(p\cdot x + q\cdot y + r\cdot z)} \equiv \ldots$$

Since m_R^2 should stay finite, we would like to reach $(am_R)^2 \to 0$ [continuum limit] at $\lambda_R \neq 0$ [non-trivial].



Does λ_R remain finite as $am_R \to 0$? Exercise!

Very enjoyable reading: M. Lüscher, P. Weisz,
Nucl. Phys. B 290 (1987) 25; 295 (1988) 65; 300 (1988) 325; 318 (1989) 705.

Analytic approximation schemes

“Weak coupling”: expand in $\lambda_0 \phi_0^4$.
[+ works also in continuum − only asymptotic series].

“Strong coupling”: expand in $\sum_\mu \phi_0(x)\phi_0(x + a\hat{\mu})$.
[+ finite radius of convergence − not in continuum].

Numerical methods

Monte Carlo integration.
[$\int_0^1 \mathrm{d}x\, f(x) \approx \frac{1}{N}\sum_{i=1}^N f(x_i)$, x_i random numbers;
superior to classical algorithms if dimensionality $\gtrsim 10$]

Importance sampling.
[Take more values where f large, but need $f \geq 0$]

Professional error estimation.

General literature

I. Montvay and G. Münster,
Quantum Fields on a Lattice
(Cambridge University Press, 1994).

H.J. Rothe,
Lattice Gauge Theories: An Introduction
(World Scientific, 2005).

J. Smit,
Introduction to Quantum Fields on a Lattice
(Cambridge University Press, 2002).

Lattice exercise set 1

1. Inserting

$$\phi_0(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{\phi}_0(p) e^{ip\cdot x}$$

into

$$S_E^{(2)} = a^4 \sum_x \left\{ \frac{1}{2} \sum_\mu (\Delta_\mu \phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 \right\} ,$$

derive the scalar propagator in lattice regularization.
[Hint: $a^4 \sum_x e^{ip\cdot x} = (2\pi)^4 \Pi_{\mu=0}^3 \delta(p_\mu \bmod \frac{2\pi}{a})$.]

2. The following equation can be derived in weak-coupling expansion (given some definition of λ_R):

$$am_R \left.\frac{\partial \lambda_R}{\partial (am_R)}\right|_{\lambda_0} = \frac{9}{8\pi^2}\lambda_R^2 + \mathcal{O}(\lambda_R^3) \, .$$

How does λ_R behave for $am_R \to 0$? What do you conclude from here?

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Lecture 2

The perfect field theory: discretised pure SU(N_{c}).

The continuum SU(N_c) Yang-Mills Lagrangian:

$$
\begin{aligned}
\mathcal{L}_M &= -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \, , \\
G^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_0 f^{abc} A^b_\mu A^c_\nu \, .
\end{aligned}
$$

Or:

$$
\begin{aligned}
\mathcal{L}_M &= \frac{1}{2g_0^2} \mathrm{Tr}\{[D_\mu, D_\nu][D^\mu, D^\nu]\} \, , \\
D_\mu &= \partial_\mu + i g_0 A^a_\mu T^a \, ,
\end{aligned}
$$

where T^a are traceless Hermitean $N_\mathrm{c} \times N_\mathrm{c}$-matrices, satisfying $[T^a, T^b] = i f^{abc} T^c$ and $\mathrm{Tr}[T^a T^b] = \delta^{ab}/2$.

The principle behind this structure:
local gauge invariance.

$$A_\mu(x) \equiv A_\mu^a(x) T^a \longrightarrow$$

$$A'_\mu(x) = G(x) A_\mu(x) G^{-1}(x) - \frac{i}{g_0} G(x) \partial_\mu G^{-1}(x) \ ,$$

$$D_\mu(x) \longrightarrow D'_\mu(x) = G(x) D_\mu(x) G^{-1}(x) \ ,$$

where $G(x) \in \mathrm{SU}(N_\mathrm{c})$ is an arbitrary differentiable fcn.

That ∂_μ appears in the transformation of A_μ means that this transformation is somehow “non-local”.

Analytic continuation:

$$it = \tau \ , \quad \partial_t = i\partial_\tau \ , \quad A_0^M = iA_0^E \ .$$

[Might justify by: take gauge $A_0^M = 0$, introduce A_0^E by writing the Gauss law $\delta(G)$ as $\int \mathrm{d}A_0^E \exp(iA_0^E G)$.]

$$\begin{aligned} \Longrightarrow \qquad & G_{ij}^M = G^{Mij} = G_{ij}^E \ , \\ & G_{0i}^M = iG_{0i}^E \ , \\ & G^{M0i} = -iG_{0i}^E \ , \\ & \mathcal{L}_E = -\mathcal{L}_M = \frac{1}{4} G_{\mu\nu}^{Ea} G_{\mu\nu}^{Ea} \ . \end{aligned}$$

We drop the superscripts E in the following.

Wilson's idea: replace the variable $A_\mu(x)$ by a **link** $U_\mu(x) \in \mathrm{SU}(N_\mathrm{c})$, which has a directionality like $A_\mu(x)$:

$$x \xrightarrow{U_\mu} x + a\hat{\mu}$$

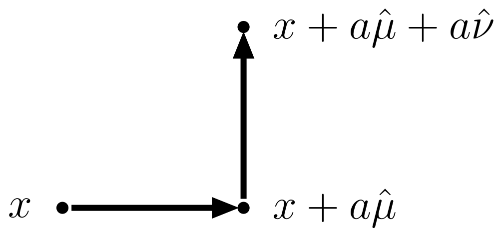
This may be called a "parallel transporter". Backwards: $U_\mu^{-1}(x) = U_\mu^\dagger(x)$. Gauge transformation:

$$U_\mu(x) \to U'_\mu(x) = G(x)U_\mu(x)G^{-1}(x + a\hat{\mu}) \ .$$

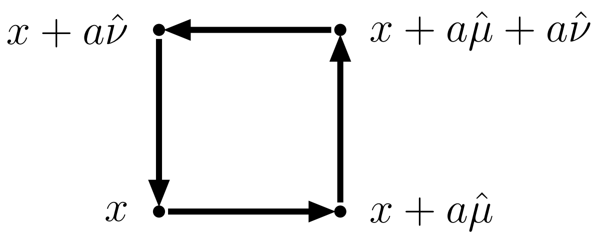
The arrow should actually point to the left!

A product of two links transforms as

$$U_\mu(x)U_\nu(x+a\hat{\mu})\rightarrow G(x)U_\mu(x)U_\nu(x+a\hat{\mu})G^{-1}(x+a\hat{\mu}+a\hat{\nu}) \ .$$



Therefore we obtain an invariant quantity by building a loop, or a *plaquette* $P_{\mu\nu}(x)$, with $P_{\mu\nu}(x) \rightarrow G(x)P_{\mu\nu}(x)G^{-1}(x)$, so that $\mathrm{Tr}[P_{\mu\nu}(x)]$ is invariant.



Requiring furthermore rotational symmetry, we obtain

$$S_E \equiv \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu=0}^{3} \mathrm{Tr}\left[\mathbb{1} - P_{\mu\nu}(x)\right] \ ,$$

where $\mathbb{1}$ is a convention. No place for a^4 here!

K.G. Wilson, Phys. Rev. D 10 (1974) 2445.

Let us see if it gives the correct continuum limit.

$$
\begin{aligned}
U_\mu(x) &\equiv e^{iag_0 A_\mu(x)} \,, \\
P_{\mu\nu}(x) &\approx e^{iag_0[A_\mu(x)+A_\nu(x+a\hat{\mu})-A_\mu(x+a\hat{\nu})-A_\nu(x)]} \\
&= e^{ia^2 g_0\left[\Delta_\mu A_\nu(x)-\Delta_\nu A_\mu(x)\right]} \\
&\approx e^{ia^2 g_0\, G^a_{\mu\nu}T^a} \,, \\
\mathrm{Tr}[P_{\mu\nu}] &\approx \mathrm{Tr}[\mathbb{1}] - \frac{1}{2}a^4 g_0^2\,\mathrm{Tr}\left[(G^a_{\mu\nu}T^a)^2\right] \,, \\
S_E &\approx a^4 \sum_x \frac{1}{4} G^a_{\mu\nu}G^a_{\mu\nu} \,. \qquad \text{OK!}
\end{aligned}
$$

Works even to 4^{th} order in A^a_μ: H.J. Rothe, *Lattice Gauge Theories: An Introduction*.

We also need to define a finite and invariant integration measure. This is a well-understood topic in group theory: requiring that $\int \mathrm{d}U \equiv \int_U$ is

- linear: $\int_U \sum_i \lambda_i f_i(U) = \sum_i \lambda_i \int_U f_i(U)$;
- positive: $f(U) > 0 \Rightarrow \int_U f(U) > 0$;
- invariant: $\int_U f(U) = \int_U f(UU') = \int_U f(U'U)$,
- normalised: $f(U) = 1 \Rightarrow \int_U f(U) = 1$;

produces (essentially uniquely) the **Haar measure.**

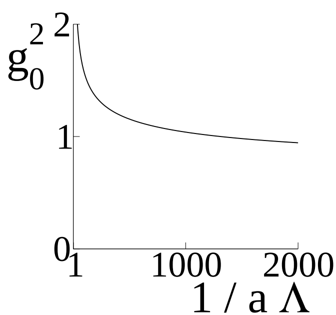
Note that integrals are finite: no gauge fixing needed! (But could be introduced, in a well-defined way.)

Renormalization

There is a single parameter, the bare coupling g_0^2. We denote a renormalised coupling by g_R^2; its definition is, again, not unique. Weak-coupling expansion produces

$$a\,\frac{\partial g_0^2}{\partial a}\bigg|_{g_R^2} = \frac{11N_\mathrm{c}}{24\pi^2}g_0^4 + \mathcal{O}(g_0^6) \;\Rightarrow g_0^2 \approx \frac{24\pi^2}{11N_\mathrm{c}\ln(1/a\Lambda)}\,,$$

where Λ is a so-far unknown constant.



A. Hasenfratz, P. Hasenfratz, Phys. Lett. B 93 (1980) 165.

Thus, we can keep physics non-trivial ($g_R^2 > 0$) and simultaneously approach the continuum limit, simply by tuning $g_0^2 \to 0$! The sign in the RG-running allowing for this is a manifestation of **asymptotic freedom**.

For reference: one often denotes $\beta_{\mathsf{G}} \equiv 2N_{\rm c}/g_0^2$. For $N_{\rm c} = 3$, $\beta_{\mathsf{G}} = 6.0$ (i.e. $g_0^2 = 1.0$) is found to correspond to $a \approx 0.1$ fm, by computing certain physical observables (see below). This fixes Λ.

Observables

Suppose I is not gauge-invariant: $I[U'] = G(x)I[U]$.
Using the invariances of measure and action,

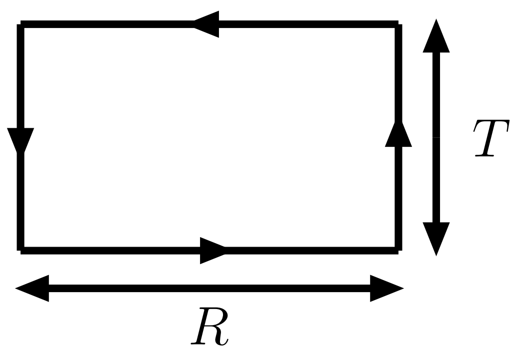
$$\begin{aligned} \langle I \rangle &= \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, I[U] e^{-S_E[U]} \\ &= \frac{1}{\mathcal{Z}} \int \mathcal{D}U \, I[U'] e^{-S_E[U]} = G(x) \langle I \rangle \, . \end{aligned}$$

Take $G(x)$ from the center $\mathbb{Z}_{N_c} = \{ e^{\frac{2\pi i n}{N_c}} \mathbb{1}_{N_c \times N_c} \}$.

$$\Rightarrow \langle I \rangle = e^{\frac{2\pi i n}{N_c}} \langle I \rangle = \frac{1}{N_c} \sum_{n=1}^{N_c} e^{\frac{2\pi i n}{N_c}} \langle I \rangle = 0 \, .$$

Thus gauge non-invariant observables vanish!

A prime example of a gauge-invariant observable: the Wilson loop.



$$W(R,T) \equiv \frac{1}{N_{\mathrm{c}}} \mathrm{Tr} \left[\prod_{\mathsf{links}} U_\mu \right] .$$

Now we define a **static potential**:

$$V(R) \equiv - \lim_{T\to\infty} \frac{1}{T} \ln\langle W(R,T)\rangle \ .$$

Interpretation: a static "quark" and "anti-quark" sit a distance R apart and "propagate" in time. The expectation value for this is

$$\langle W(R,T)\rangle \approx C \exp\left(-TV(R)\right) \ ,$$

where $V(R)$ is the ground-state energy of such a configuration.

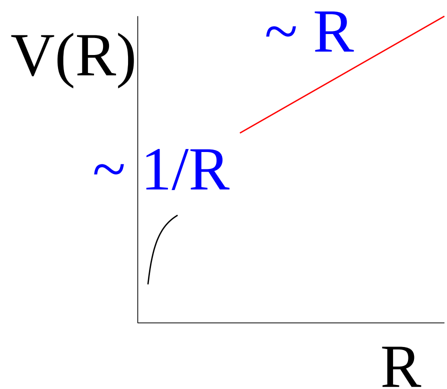
$V(R)$ can be computed in various limits.

Weak-coupling expansion $g_0^2 \ll 1$ [exercise]:

$$V(R) \approx -\frac{N_c^2 - 1}{2N_c}\frac{g_0^2}{4\pi R}\,.$$

Strong-coupling expansion $1/g_0^2 \ll 1$ [tough exercise]:

$$V(R) \approx \frac{R}{a^2}\ln(g_0^2 N_c)\,.$$



Continuum limit: $g_0^2 \to 0 \Rightarrow$ strong-coupling expansion not justified. But numerical results show that the linear rise persists! This supposedly implies **confinement**, since such a $V(R)$ only allows for bound state solutions. Coefficient: $V(R) = \sigma R$, $\sigma \equiv$ **string tension**.

We write $\sqrt{\sigma} \approx 470$ MeV $\sim E_{\mathrm{QCD}} \equiv$ "QCD scale" .

See e.g. M. Teper, hep-th/9812187.

Another manifestation of confinement:

$$\left\langle \int_{\mathbf{x},\mathbf{y}} \mathrm{Tr}[P_{\mu\nu}](\tau,\mathbf{x})\,\mathrm{Tr}[P_{\alpha\beta}](0,\mathbf{y}) \right\rangle \propto e^{-m_{\text{glueball}}\tau} \; ,$$

where $m_{\text{glueball}} \approx 1700$ MeV is the “glueball mass”. In other words, the pure gauge theory has a **mass gap**.

(Prove this mathematically, and get some $ from `http://www.claymath.org/millennium/`)

Lattice exercise set 2

1. Show that in the weak-coupling expansion, the static potential of pure SU(N_c) gauge theory reads

$$V(R) = -\frac{N_\mathrm{c}^2 - 1}{2N_\mathrm{c}} \frac{g_0^2}{4\pi R} + \text{constant} + \mathcal{O}\left(\frac{a}{R^2}\right) \; .$$

[Hint: You can use continuum propagators since the leading-order term is a-independent; the "horizontal" legs of the Wilson loop do not contribute at this order; you can use Feynman gauge since the observable is manifestly gauge-invariant.]

2. Compute the expectation values $\langle U_\mu(x)\rangle$, $\langle U_\mu(x)U_\nu(y)\rangle$, and $\langle U_\mu(x)U_\nu^\dagger(y)\rangle$ to leading order in the strong-coupling expansion. Can you give a graphical interpretation to these results?

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Lecture 3

What should we like to do for the real world?

Different theories:

Electromagnetic interactions: $\alpha_{\mathsf{em}} \approx \frac{1}{137}$
$\Rightarrow$ Perturbation theory works extremely well.

Weak interactions: $\alpha_{\mathsf{w}} \approx \frac{1}{30}$
$\Rightarrow$ Perturbation theory works pretty well.

Strong interactions: $\alpha_{\mathsf{s}} \approx \frac{1}{10} \ldots 1$
$\Rightarrow$ Perturbation theory works in some cases, others not.

Gravitational interactions: ?

So we will concentrate on QCD here.

It can be noted, though, that at finite temperatures even electromagnetic and weak interactions can become non-perturbative, since new expansion parameters, $\sim \alpha T/m_{\text{eff}}(T)$, get generated.

[Expand $\alpha n_{\text{b}}(m_{\text{eff}}) = \alpha/(e^{m_{\text{eff}}/T} - 1)$ for small m_{eff}/T!]

Around **phase transitions** $m_{\text{eff}}(T)$ can be very small (first order) or even vanish (second order). Therefore the properties of the phase transition where the Higgs mechanism sets in, as well as those of the superconducting phase transition, are non-trivial.

The quark part of the QCD Lagrangian:

$$\delta\mathcal{L}_M = \bar{\psi}\left[i\gamma^\mu D_\mu - M\right]\psi \ .$$

To be explicit:

$$\begin{aligned}
\bar{\psi} &= \bar{\psi}_{i\alpha A}(x)\ , \quad \psi \ = \ \psi_{j\beta B}(y)\ , \\
x, y &= \text{spacetime coordinates}\ , \\
i, j &= 1, \ldots, N_\mathrm{f} = \text{flavour indices}\ , \\
\alpha, \beta &= 1, \ldots, 4 = \text{spinor (Dirac) indices}\ , \\
A, B &= 1, \ldots, N_\mathrm{c} = \text{colour indices}\ , \\
\gamma^\mu D_\mu &= \delta_{ij}[\gamma^\mu]_{\alpha\beta}\left(\partial_\mu \delta_{AB} + i g_0 A^a_\mu T^a_{AB}\right)\ , \\
M &= \mathrm{diag}(m_u, m_d, m_s, m_c, m_b, m_t)_{ij}\delta_{\alpha\beta}\delta_{AB}\ .
\end{aligned}$$

Analytic continuation:

$$
\begin{aligned}
\partial_t &= i\partial_\tau \ , \\
D_t &= iD_\tau \ , \\
\gamma^i &\equiv i\gamma_i^E \ , \\
\gamma^0 &\equiv \gamma_0^E \ , \\
\mathcal{L}_E &= -\mathcal{L}_M \\
\Rightarrow \delta S_E &= \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x}\, \bar{\psi}\left[\gamma_\mu^E D_\mu + M\right]\psi \ .
\end{aligned}
$$

We drop the superscript from γ_μ^E in the following.

Discretisation

We return to the details of the discretisation of D in the next lecture, and for the moment simply write

$$\delta S_E = a^4 \sum_{x,y} \bar{\psi}(x) \left[D(x,y) + M\delta_{x,y} \right] \psi(y) \, .$$

Here $D(x,y)$ is called the massless Dirac operator. It is flavour diagonal, but a matrix in spinor, colour, and spacetime indices.

Now, whatever the properties of D, the action is quadratic in the Grassmann-valued fields $\bar{\psi}, \psi$. Thus fermions can be integrated out analytically. With the usual SU(N_c) action $S_E^{(\text{gluons})}$, we get

$$\mathcal{Z} = \int \mathcal{D}U_\mu \; \text{Det}[D + M] \exp\left\{-S_E^{(\text{gluons})}\right\} ,$$

where $\text{Det}[D + M]$ is gauge-invariant. Moreover,

$$\begin{aligned}&\langle \psi(x)\bar{\psi}(y)\rangle \\ &= \frac{1}{\mathcal{Z}} \int \mathcal{D}U_\mu \; \text{Det}[D + M][D + M]^{-1}(x, y) e^{-S_E^{(\text{gluons})}} .\end{aligned}$$

Here

$$\mathrm{Det}[D+M] = \prod_i \Lambda_i$$

and [exercise]

$$[D+M]^{-1}(x,y) = \sum_i \frac{v_i(x)v_i^\dagger(y)}{\Lambda_i} \ ,$$

where Λ_i and $v_i(x)$ are the eigenvalues and eigenfunctions of the matrix $[D+M](x,y)$.[2]

[2] Another way: $[D+M](x,z)\{[D+M]^{-1}(z,y)\} = \delta_{x,y} \Rightarrow$ solve the difference equation for $[D+M]^{-1}(z,y)$.

So we are left with a bosonic integral!
[Note that D and thus Λ_i depend on U_μ.]
But it is more complicated than it first appears:

• Determinant: all eigenvalues contribute on equal footing, and there are a lot of them in large volumes.

• Inversion: gauge configurations producing small eigenvalues Λ_i can give a very large contribution $\Rightarrow$ need to be very precise and control fluctuations.

A significant (ad hoc) simplification:

$\text{Det}[D+M] \equiv 1 \Leftrightarrow$ "Quenched approximation" .

$([D+M]^{-1}$, or "valence quarks", can still be kept.)

The evaluation of this integral poses many different types of challenges, which occupy the bulk of the lattice community [hep-lat: ~ 500 submissions / year].

A. Technical challenges

Algorithms

- How to compute efficiently (= fast, precisely) the eigenvalues and eigenfunctions of the (sparse) $[4N_{\rm c}N_{\rm f}\prod_{\mu=0}^{3}N_\mu] \times [4N_{\rm c}N_{\rm f}\prod_{\mu=0}^{3}N_\mu]$ Dirac operator?

- How to compute the determinant, i.e. "unquench"?

- How to optimise the importance sampling?
- How to parallelise the code efficiently?
- Etc etc.

Machines

- Does it make sense to build special purpose computers for QCD? [QCDOC, ApeNEXT].
- How to best make use of commercial PC-clusters?

B. Theoretical challenges

- How to construct a Dirac operator which respects as many continuum symmetries as possible?

$\Rightarrow$ Ginsparg-Wilson fermions.

- How to approach the continuum limit as fast as possible?

$\Rightarrow$ Improved actions.

- What kind of observables to consider / measurements to carry out? I.e., **asking the right questions**: lattice allows to probe QCD in ways not possible in accelerator experiments, and thus learn new things about it in unexpected ways.

- Can one find improved analytic approximation schemes?

- How to automatise the existing analytic methods (weak-coupling and strong-coupling expansion) such that they can be pushed to arbitrarily high order?

- Etc etc.

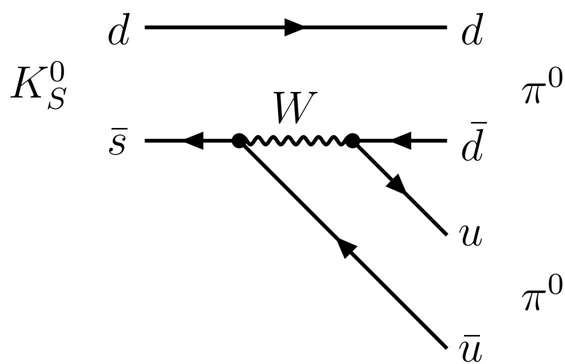
C. Phenomenological challenges

1. Is QCD really the correct theory of strong interactions? Quarks and gluons are not observed directly, but does the action written in terms of them still reproduce the properties of mesons, proton, neutron, ... ?

2. If QCD is correct, we should determine accurately its parameters (quark masses) by matching meson, proton, neutron etc masses to observed values.

3. Is Weinberg-Salam the correct theory of weak interactions? Even though weak interactions are themselves perturbative, the initial and final states may be hadrons, which can only be described by QCD! Thus QCD plays an essential role.

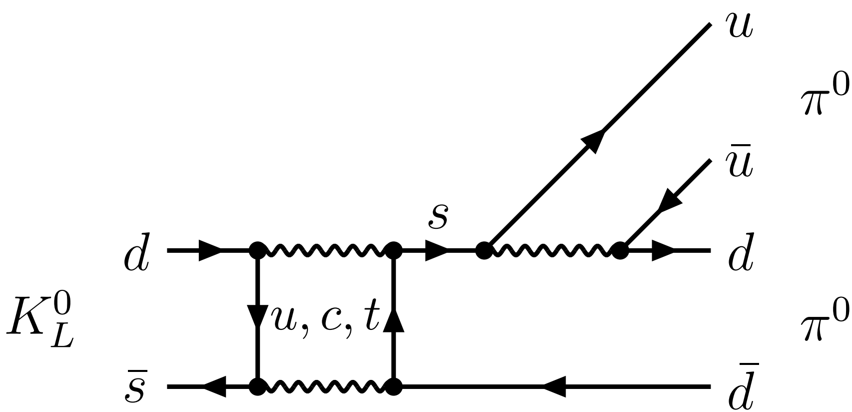
The simplest non-trivial case:



CP-even decay

Even more important: can the CP-violation in the CKM-matrix explain the amplitude of $K_L^0 \to \pi\pi$?

Schematically:



CP-odd decay

4. How do quarks and gluons behave at the extremely high temperatures relevant for the Early Universe?

   No direct experimental probes [although heavy ion collision experiments attempt to say something], lattice may be the only way!

   Needed e.g. for certain dark matter computations in cosmology.

Hindmarsh, Philipsen, hep-ph/0501232;
Asaka et al, hep-ph/0605209.

5. How do quarks and gluons behave at the extremely high densities relevant for the cores of neutron stars?

   Again no direct experimental probes, lattice may be the only way!

   Needed for understanding properties like the cooling rate and radius vs mass relationship of neutron stars.

Ozel, astro-ph/0605106;
Alford et al, astro-ph/0606524.

6. Can we **understand** quark confinement?

Although a full "understanding" would have to be analytic, lattice can be used for determining observables which are not measurable in experiment, yet reflect confining dynamics in a clean way.

For instance, the static potential for $M \to \infty$ (i.e. $N_\mathrm{f} = 0$) has been studied in great detail, and matched to effective string theory, to learn under which circumstances and which of the possible string theories might work here.

M. Lüscher, P. Weisz, hep-lat/0207003;
K.J. Juge, J. Kuti, C. Morningstar, hep-lat/0207004.

Lattice exercise set 3

1.a. The Minkowskian γ-matrices have the properties $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, where $\eta = \mathrm{diag}(+---)$. Show that the Euclidean γ-matrices satisfy $\{\gamma^E_\mu, \gamma^E_\nu\} = 2\delta_{\mu\nu}$, $(\gamma^E_\mu)^\dagger = \gamma^E_\mu$.

1.b. We define $\gamma_5 \equiv \gamma^E_0\gamma^E_1\gamma^E_2\gamma^E_3$. Show that it has the properties

$$\{\gamma_5, \gamma^E_\mu\} = 0\,, \quad \gamma_5^\dagger = \gamma_5\,, \quad \gamma_5^2 = 1\,.$$

2. Let us consider a single quark flavour with mass m, so that the Dirac operator is $D+m$, where D is the massless Dirac operator. Show that

$$[D+m]^{-1}(x,y) = \sum_i \frac{v_i(x)v_i^\dagger(y)}{\lambda_i+m} \,,$$

where $\lambda_i, v_i(x)$ are the eigenvalues and functions of D.

[As an extra: Show that the proper normalization of the eigenfunctions $v_i(x)$ reads $\sum_x v_i^\dagger(x)v_i(x) = 1$.]

Lattice Field Theory

Mikko Laine (University of Bielefeld, Germany)

1. Why and how? Scalar $\lambda\phi^4$ on the lattice.
2. The perfect field theory: discretised pure SU(N_c).
3. What should we like to do for the real world?
4. Theoretical challenge: chiral symmetry.
5. Practical challenge: hadron spectra from QCD.

Lecture 4

Theoretical challenge: Chiral symmetry.

Let us now study more precisely the massless Dirac operator D. We need its eigenvalues and eigenfunctions to obtain Det$[D + m]$ and $[D + m]^{-1}(x, y)$.

Recall: in continuum,

$$D = \gamma_\mu \left[\partial_\mu + i g_0 A_\mu\right] \ .$$

This is anti-Hermitean $\Rightarrow$ eigenvalues purely imaginary.

In fact we prefer to discuss "γ_5-Hermiticity", $D^\dagger = \gamma_5 D \gamma_5$, since this is not affected by the mass term.

Discretisation

Keeping in mind that $U_\mu \approx \mathbb{1} + iag_0 A_\mu$ near the continuum limit and that in gauge transformations $U_\mu(x) \to G(x)U_\mu(x)G^{-1}(x+a\hat{\mu})$, let us first try

$$
\begin{aligned}
D &= \gamma_\mu D_\mu(x,y) \; , \\
D_\mu(x,y) &= \frac{1}{a}\left[U_\mu(x)\delta_{y,x+a\hat{\mu}} - \delta_{y,x}\right] \; .
\end{aligned}
$$

This does not look optimal, however, since the matrix is not γ_5-Hermitean: $D_\mu^\dagger(y,x) \neq -D_\mu(x,y)$.

In order to make it γ_5-Hermitean, let us "symmetrise" the first trial:

$$
\begin{aligned}
D' &= \gamma_\mu D'_\mu \, , \\
D'_\mu &= \frac{1}{2a}\left[U_\mu(x)\delta_{y,x+a\hat{\mu}} - U^\dagger_\mu(x-a\hat{\mu})\delta_{y,x-a\hat{\mu}}\right] \\
&= \frac{1}{2a}\left[U_\mu(x)\delta_{y,x+a\hat{\mu}} - U^\dagger_\mu(y)\delta_{x,y+a\hat{\mu}}\right] \, .
\end{aligned}
$$

Now $D'^\dagger(y,x) = \gamma_5 D'(x,y)\gamma_5$.

Let us look at the free propagator that this leads to. Writing

$$\psi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \tilde{\psi}(p)\, e^{ip\cdot x} \;,$$

$$\bar{\psi}(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \tilde{\bar{\psi}}(p)\, e^{-ip\cdot x} \;,$$

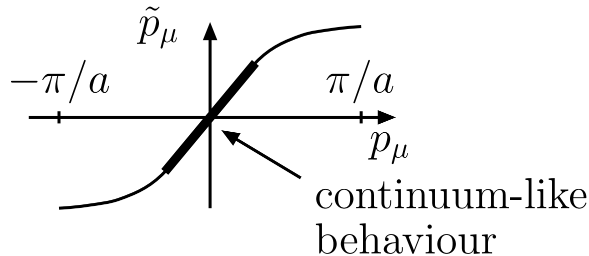
we find [exercise]

$$\delta S_E = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \tilde{\bar{\psi}}(p)\; [i\gamma_\mu \mathring{p}_\mu + m]\, \tilde{\psi}(p) \;,$$

where $\mathring{p}_\mu \equiv \frac{1}{a} \sin a p_\mu$.

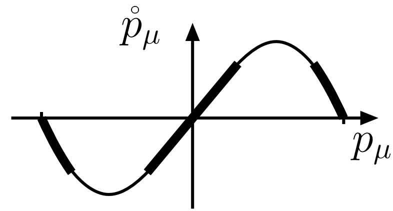
Integration range: $p_\mu \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right)$.

Bosonic case: $\tilde{p}_\mu = \frac{2}{a} \sin \frac{a p_\mu}{2}$.



So $\tilde{p}_\mu$ vanishes only around the origin.

Fermionic case: $\mathring{p}_\mu \equiv \frac{1}{a}\sin ap_\mu$.



There are two regions with a continuum-like behaviour! This is the fermion **doubling problem**. In four dimensions, our naive D' describes 2^4 light fermions, rather than one as we wanted.

The idea of the so-called **Wilson discretisation** is that we can modify the naive discretisation by $\mathcal{O}(a)$ effects, and thus try to get rid of the doubling problem.

In particular, let us add the term

$$\delta S_E = -\frac{r}{2} \sum_x a^5 \bar{\psi}(x) \Delta_\mu \Delta_\mu^* \psi(x) \ ,$$

where $r > 0$ is a free parameter.

The denominator of a propagator now reads [exercise]:

$$\mathring{p}^2 + \frac{a^2 r^2}{4}\tilde{p}^4 + m(m + ar\tilde{p}^2)\ .$$

Thus the mass-independent part has been lifted at $p_\mu = \pm\pi/a$ by the large amount $\sim r^2/a^2$, and the problem seems solved.

There is a price to pay, however: chiral symmetry has been lost. What is chiral symmetry?

Define a global "vector" transformation $\in \mathrm{U}_V(1)$,

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{-i\alpha} \,, \quad \psi \to \psi' = e^{i\alpha} \psi \,,$$

as well as a global "axial" transformation $\in \mathrm{U}_A(1)$,

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{i\alpha\gamma_5} \,, \quad \psi \to \psi' = e^{i\alpha\gamma_5} \psi \,.$$

[One might recall the 3rd one from $\psi^\dagger e^{-i\alpha\gamma_5}\gamma_0 = \psi^\dagger \gamma_0 e^{i\alpha\gamma_5}$, although it is really a definition, since $\bar{\psi}, \psi$ are independent integration variables.]

The term $\bar{\psi}\gamma_\mu D_\mu \psi$ is invariant in both transformations, since $\{\gamma_\mu, \gamma_5\} = 0$, so that $\{\gamma_\mu D_\mu, \gamma_5\} = 0$.

The mass term $\bar{\psi} M \psi$ is also invariant in $\mathrm{U}_V(1)$.

But $\bar{\psi} M \psi$ is not invariant in $\mathrm{U}_A(1)$.

Both symmetries appear, however, to be recovered in the **chiral limit** $M \to 0$!

The problem now is that the new Wilson term is of the type $\bar{\psi}\psi$ even for $M \to 0$! Thus it appears that we have lost the U$_A(1)$ symmetry of the chiral limit.

It is believed that the symmetry can still be recovered by tuning the mass matrix M to some non-zero value, but this requires a fair amount of fine-tuning.

In other words, the bare mass develops additive divergences with Wilson fermions, something that could not happen if the chiral symmetry were exact.

These problems are in fact generic, and can be expressed through the Nielsen-Ninomiya theorem, saying that the following cannot hold simultaneously:

1. The Fourier transform $\tilde{D}(p)$ is analytic [“locality”].
2. For $|p| \ll \pi/a$, $\tilde{D}(p) = i\gamma_\mu p_\mu + \mathcal{O}(ap^2)$.
3. There are no doublers [$\tilde{D}(p) = 0$ only for $p = 0$].
4. There is chiral [$\mathrm{U}_A(1)$] symmetry: $\gamma_5 D + D\gamma_5 = 0$.

The deep reason behind this: if all of this were true, we could have a well-defined chirally symmetric theory with a single fermion. But we know that this cannot be the case: **chiral anomaly** has to appear somewhere!

Why is the loss of chiral symmetry a problem?

- The presence of ultraviolet divergences which were previously guaranteed to be absent by chiral symmetry (like additive mass divergences) complicate the renormalization program.

- Issues related to the chiral anomaly (like Atiyah-Singer index theorem and defining topology on the lattice) cannot be rigorously addressed.

- Operators mediating weak decays develop mixings previously forbidden by the chiral symmetry — unphysical decay channels open up.

During the last few years, however, these problems have been (theoretically) completely solved!

The break-through came by looking for massless Dirac operators satisfying the so-called **Ginsparg-Wilson relation**:

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \ .$$

Note: also in continuum it is not clear what $\{\gamma_5, \gamma_\mu\}$ should be, if we regulate the theory by going from 4 to $4 - 2\epsilon$ dimensions.

There are different practical realizations of this kind of a Dirac operator, like "overlap" = "Neuberger", "domain wall", and "perfect action", but we do not need to be concerned about that here.

The idea now is to **modify what we mean by chiral symmetry** by effects of $\mathcal{O}(a)$, and show that if we use a Ginsparg-Wilson Dirac operator, such a symmetry is exact even with a finite lattice spacing!

M. Lüscher, Phys. Lett. B 428 (1998) 342.

Define the chiral/axial transformation as

$$\delta\psi = i\gamma_5(1 - aD)\psi \;, \quad \delta\bar{\psi} = \bar{\psi} i\gamma_5 \;.$$

Then [exercise] $\delta(\bar{\psi} D\psi) = 0$: U$_A$(1) is indeed there!

The Nielsen-Ninomiya theorem can now be circumvented, since its point 4 is no longer true. Thus there need not be any doublers. (There still can be, with a bad choice of D).

Moreover the chiral anomaly is recovered: the transformation of ψ involves the gauge fields through D. If we compute the Jacobian for this transformation in the integration measure $\mathcal{D}\bar{\psi}\mathcal{D}\psi$, then a corresponding term $\sim \mathrm{Tr}[\gamma_5 D]$ arises, which for smooth gauge fields can be shown to reproduce $\sim \int_{\tau,\mathbf{x}} \frac{g^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$.

Consequences

- Unwanted divergences and mixings disappear.
- Discretisation artifacts only start at $\mathcal{O}(a^2)$.
- There is an index theorem even at non-zero lattice spacing: zero-modes of D are eigenfunctions of γ_5, and $\mathrm{index}[D] \equiv n_- - n_+$, where n_- (n_+) count the zero-modes with eigenvalue -1 ($+1$), equals the classical $\sim \frac{g^2}{32\pi^2}\int_{\tau,\mathbf{x}} G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$ for smooth gauge fields.
- One can define a topological susceptibility which has a finite yet non-trivial continuum limit.

Of course there is a price to pay, namely that this Dirac operator is very expensive to simulate in practice.

Some reviews:

M. Lüscher,
“Chiral Gauge Theories Revisited”,
hep-th/0102028.

S. Chandrasekharan and U.-J. Wiese,
“An Introduction to Chiral Symmetry on the Lattice”,
hep-lat/0405024.

P. Hasenfratz,
“Chiral Symmetry on the Lattice”,
hep-lat/0406033.

Lattice exercise set 4

1. Show that the free propagator for Wilson fermions reads

$$\frac{-i\gamma_\mu \mathring{p}_\mu + m + \frac{1}{2}ar\tilde{p}^2}{\mathring{p}^2 + (m + \frac{1}{2}ar\tilde{p}^2)^2} \ ,$$

where

$$\mathring{p}^2 = \sum_\mu \mathring{p}_\mu^2 \ , \quad \mathring{p}_\mu = \frac{1}{a}\sin ap_\mu \ ,$$

$$\tilde{p}^2 = \sum_\mu \tilde{p}_\mu^2 \ , \quad \tilde{p}_\mu = \frac{2}{a}\sin \frac{ap_\mu}{2} \ .$$

[The free propagator of the naive discretisation is obtained by setting $r \to 0$.]

2. Show that if the Dirac operator D respects the Ginsparg-Wilson relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$, and we define the infinitesimal axial transformations as $\delta\psi = i\gamma_5(1 - aD)\psi,\ \delta\bar{\psi} = \bar{\psi}i\gamma_5,$ then $\delta(\bar{\psi}D\psi) = 0$.

Lattice Field Theory

Mikko Laine (University of Bielefeld, Germany)

1. Why and how? Scalar $\lambda\phi^4$ on the lattice.
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Lecture 5

Practical challenge:
Hadron spectra from QCD.

Recall pure SU(3): there exists a "QCD scale" (determining string tension, glueball mass, etc) of a few hundred MeV (definition and thus numerical value vary). Let us call this E_{QCD}.

Light quarks:

$$m_u, m_d, m_s \ll E_{\text{QCD}} \ .$$

Heavy quarks:

$$m_c, m_b, m_t \gg E_{\text{QCD}} \ .$$

In the following we consider hadrons ($=$ mesons, baryons) made out of the light quarks.

Lightest mesons [the eight-fold way]:

particle	mass	valence quark content
π^0	135 MeV	$u\bar{u} - d\bar{d}$
$\pi^\pm$	140 MeV	$u\bar{d}, d\bar{u}$
$K^0, \bar{K}^0$	498 MeV	$d\bar{s}, s\bar{d}$
$K^\pm$	494 MeV	$u\bar{s}, s\bar{u}$
η	547 MeV	$u\bar{u} + d\bar{d} - 2s\bar{s}$

Lightest baryons:

particle	mass	valence quark content
p	938 MeV	uud
n	940 MeV	udd

Can we deduce m_u, m_d, m_s from these values?

Consider the pion. In principle: take any "interpolating operator" with the correct quantum numbers, e.g.

$$\Pi^0 = Z_\pi \bar{\psi} i\gamma_5 T^3 \psi \ , \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} ;$$

take a lattice with a large τ-extent β; and measure

$$\int_{\mathbf{x},\mathbf{y}} \langle \Pi^0(\tau,\mathbf{x}) \Pi^0(0,\mathbf{y}) \rangle = C \exp(-m_\pi \tau) + \ldots$$

There is, however, a problem in doing this: in order for discretisation errors to be small, lattice spacing should be smaller than the Compton wavelength of the heaviest relevant excitation, say the glueball:

$$a \ll \lambda_{\mathsf{glueball}} = \frac{2\pi}{1.7\ \mathsf{GeV}} \approx 0.7\ \mathsf{fm}\ .$$

On the other hand, in order to see the exponential decay, lattice should be longer than the Compton wavelength of the pion:

$$\beta = Na \gg \lambda_{\mathsf{pion}} = \frac{2\pi}{0.14\ \mathsf{GeV}} \approx 9.0\ \mathsf{fm}\ .$$

So require $N \gg 100$. This is very expensive!

Another way to see the problem: quark propagator is

$$[D+m]^{-1}(x,y) = \sum_i \frac{v_i(x)v_i^\dagger(y)}{\lambda_i + m} \, ,$$

where λ_i are eigenvalues of the massless Dirac operator.

If $m \to 0$, λ_i is "not shielded", and $1/|\lambda_i|$ can fluctuate strongly between various gauge field configurations $\Rightarrow$ "bad signal" (large statistical fluctuations).

So a **scale hierarchy** ($\lambda_{\text{glueball}} \ll \lambda_{\text{pion}}$) is a serious problem for numerics.

But it also contains in itself the ingredients for a more clever way to go forward.

Indeed, systems with a scale hierarchy usually allow for a description of their low-energy / low-momentum / long-distance / infrared (IR) dynamics through **effective field theories**.

So the idea now is:

Construct an effective field theory describing the IR dynamics.

Use the effective field theory for addressing physical observables.

Use lattice only for determining the **parameters** of this effective field theory! (I.e. not physics directly.)

The reason for the low meson masses is that they are the almost Goldstone bosons of the spontaneously broken flavour symmetry, $\mathsf{SU}_L(N_\mathrm{f})\times\mathsf{SU}_R(N_\mathrm{f}) \rightarrow \mathsf{SU}_V(N_\mathrm{f})$ [only "almost" because $M \neq 0$].

We describe these $N_\mathrm{f}^2 - 1$ Goldstone bosons as fluctuations $\omega^I(x)$ around the minimum $U \equiv \mathbb{1}$ of the Goldstone manifold:

$$\begin{aligned} U(x) &= \mathbb{1} + i \sum_{I=1}^{N_\mathrm{f}^2-1} \omega^I(x) T^I + \mathcal{O}(\omega^2) \\ &\equiv \exp(i \sum_{I=1}^{N_\mathrm{f}^2-1} \omega^I(x) T^I) \in \mathsf{SU}(N_\mathrm{f}) \, , \quad N_\mathrm{f} = 3 \, . \end{aligned}$$

The effective chiral Lagrangian is now the most general Lagrangian which respects the original flavour symmetry of the system $[U \to G_L U G_R^\dagger]$ and is consistent with the ground state being at $U \approx \mathbb{1}$:

$$
\begin{aligned}
S_E &= \int_{\tau,\mathbf{x}} \mathrm{Tr}\Big\{ \tfrac{F^2}{4} \partial_\mu U \partial_\mu U^\dagger - \tfrac{\Sigma}{2}\big[U M^\dagger + M U^\dagger \big] \Big\} + S_E^{\text{h.o.}} \; , \\
S_E^{\text{h.o.}} &= \int_{\tau,\mathbf{x}} \big\{ L \, \mathrm{Tr}[\partial_\mu U \partial_\mu U^\dagger] \, \mathrm{Tr}[\partial_\nu U \partial_\nu U^\dagger] + \ldots \big\} \; .
\end{aligned}
$$

The parameters appearing here are called:

$$
\begin{aligned}
F &= \text{pion decay constant (in the chiral limit)}, \\
\Sigma &= \text{chiral condensate (in the chiral limit)}, \\
L\text{'s} &= \text{higher order Gasser-Leutwyler constants} \; .
\end{aligned}
$$

The remarkable fact is that in the very chiral limit, $M \to 0$, Goldstone bosons do become massless, and the leading order chiral Lagrangian describes the dynamics **exactly**, since higher dimensional operators can only give contributions suppressed by $\mathcal{O}(m/E_{\mathsf{QCD}})$!

For $M \neq 0$, the higher order corrections are finite, but small, given that $m_u, m_d, m_s \ll E_{\mathsf{QCD}}$. In practice, NLO predictions reproduce very well the properties of the light hadron spectrum and interactions, if F, Σ and L's are tuned appropriately.

So now the goal for lattice is: determine F, Σ and the L's, rather than the physical meson masses directly.

Formally, for instance: $M = \mathrm{diag}(m_u, m_d, m_s)$,

$$
\begin{aligned}
\Sigma &= \lim_{M\to 0} \lim_{V\to\infty} \frac{1}{\beta V} \frac{\partial}{\partial m_u} \ln \mathcal{Z}_{\text{chiral}} \\
&= \lim_{M\to 0} \lim_{V\to\infty} \frac{1}{\beta V} \frac{\partial}{\partial m_u} \ln \mathcal{Z}_{\text{QCD}} \\
&= -\lim_{M\to 0} \lim_{V\to\infty} \frac{1}{\beta V} a^4 \sum_x \langle \bar{u}(x) u(x) \rangle \ .
\end{aligned}
$$

But wait: why should the simulations be easier now?

The point is that we do not actually need to take the very limits $M \to 0$, $V \to \infty$: the effective theory also tells correctly how the limits are **approached**! The only requirements are that M be small compared with E_{QCD} and the box size large compared with $\lambda_{\text{glueball}}$.

For instance, we could keep M at its physical value, and take:

$$\lambda_{\text{glueball}} \approx 0.7 \text{ fm} \Rightarrow \beta \gtrsim 2.0 \text{ fm} ,$$

rather than $\beta \gg 9.0$ fm as before.

In fact, staying in this regime, the orders of the two limits ($M \to 0$, $V \to \infty$) can even be interchanged.

As an explicit example: the chiral theory leads to the prediction $[P_L \equiv (1-\gamma_5)/2]$

$$\lim_{M\to 0} \int_{\mathbf{x}\in\mathbf{V}} \left\langle [\bar{\psi}\gamma_0 T^I P_L \psi](\tau,\mathbf{x})[\bar{\psi}\gamma_0 T^J P_L \psi](0,\mathbf{0}) \right\rangle$$

$$= \mathrm{Tr}[T^I T^J] \frac{F^2}{2\beta} \left\{ 1 + \mathcal{O}\left(\frac{1}{F^2\sqrt{\beta V}} \right) \right\} ,$$

which can be used for matching F^2 at finite volumes.

So now the hard test: match to experiment demands

$$\begin{aligned}\Sigma &\approx (250 \text{ MeV})^3 ,\\ F &\approx 87 \text{ MeV} .\end{aligned}$$

Quenched lattice results suggest something like

$$\begin{aligned}\Sigma &\simeq [(270 \pm 10) \text{ MeV}]^3 ,\\ F &\simeq (105 \pm 5) \text{ MeV} ,\end{aligned}$$

but this framework is not really well-defined.

How about the full unquenched QCD??

$$\Sigma = \ldots , \quad F = \ldots .$$

Summary

Lattice regularization is the framework for rigorous theoretical work in realistic (non-supersymmetric) gauge theories, though technically demanding.

For phenomenology, need great patience and care, but there is the potential of obtaining results which have permanent value. In this context it is important to come up with “good” questions, though.

Dynamical quarks with realistic masses are becoming feasible only now.

Keep in mind when demanding/producing fast results:

"For a successful technology, reality must take precedence over public relations, for nature cannot be fooled."

Feynman's Appendix to the Rogers Commission Report on the Space Shuttle Challenger Accident (1986).

Lattice exercise set 5

Let us assume that $M = \mathrm{diag}(m, m, m_s)$ and

$$
\begin{aligned}
U &= \exp\left(\frac{\sqrt{2}i\xi}{F}\right) , \\
\xi &\equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} , \\
S_E &= \int \mathrm{d}\tau \mathrm{d}^3\mathbf{x}\, \mathrm{Tr}\left\{ \tfrac{F^2}{4} \partial_\mu U \partial_\mu U^\dagger - \tfrac{\Sigma}{2} \left[U M^\dagger + M U^\dagger \right] \right\} .
\end{aligned}
$$

1. Show that the tree-level kaon and pion masses are

$$m_{\pi^0}^2 = m_{\pi^\pm}^2 = \frac{2m\Sigma}{F^2} \,,$$

$$m_{K^0,\bar{K}^0}^2 = m_{K^\pm}^2 = \frac{(m+m_s)\Sigma}{F^2} \,.$$

2. Use the physical values of $m_{\pi^\pm}, m_{K^\pm}$ to estimate the ratio m_s/m. Compare with the value that can be extracted from the PDG Booklet [or pdg.lbl.gov].