8.4 Relativistische Elektrodynamik

Zuletzt wollen wir alle Bausteine der Elektrodynamik — Ladungs- und Stromdichte; elektromagnetische Potentiale und Felder; Maxwell-Gleichungen; Lorentz-Kraft — in Vierer-Notation umschreiben.

4-Stromdichte

Definition: $\quad J := \begin{pmatrix} c\rho \\ \vec{j} \end{pmatrix}$

Warum ist dies ein 4-Vektor?

$$* \quad c\rho(x^0,\vec{x}) = c\sum_a q_a\, \delta^{(3)}(\vec{x}-\vec{x}_a(x^0))$$
$$= c\sum_a q_a \int dx_a^0\, \delta^{(3)}(\vec{x}-\vec{x}_a(x_a^0))\,\delta(x^0-x_a^0)$$
$$= c\sum_a q_a \int d\tau_a\, \delta^{(4)}(x-x_a(\tau_a))\,\underbrace{\frac{dx_a^0}{d\tau_a}}_{u_a^0} \quad \text{(vgl. Seite 104)}$$

$$* \quad \vec{j}(x^0,\vec{x}) = \sum_a q_a \vec{v}_a(x^0)\,\delta^{(3)}(\vec{x}-\vec{x}_a(x^0))$$
$$= \sum_a q_a \int dx_a^0\, \vec{v}_a(x_a^0)\,\delta^{(3)}(\vec{x}-\vec{x}_a(x_a^0))\,\delta(x^0-x_a^0)$$
$$= \sum_a q_a \int d\tau_a\, \delta^{(4)}(x-x_a(\tau_a))\,\underbrace{\vec{v}_a(x_a^0)\,u_a^0}_{c\cdot\vec{u}_a} \quad \text{(vgl. Seite 104)}$$

$$\Rightarrow \quad \boxed{J^\mu(x) = c\sum_a q_a \int d\tau_a\, \delta^{(4)}(x-x_a(\tau_a))\, u_a^\mu}$$

Hier:
- $d\tau_a$ ist 4-Skalar (vgl. Seite 104)
- u_a^μ ist 4-Vektor
- $\delta^{(4)}(x-x_a) = \delta^{(4)}(x'-x'_a)\,\left|\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\right|;$

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$$
$$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = \det(\Lambda)$$

Seite 106: $\Lambda\eta\Lambda^T\eta = \mathbb{1} \Rightarrow \det(\Lambda)^2 = 1 \Rightarrow |\det(\Lambda)| = +1.$

$\Rightarrow \delta^{(4)}(x-x_a) = \delta^{(4)}(x'-x'_a)$ ist 4-Skalar

$\Rightarrow$ J^μ ist 4-Vektor.

Kontinuitätsgleichung: $\quad \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = \frac{\partial(c\rho)}{\partial(ct)} + \nabla\cdot\vec{j} = \frac{\partial J^0}{\partial x^0} + \sum_i \frac{\partial J^i}{\partial x^i} = 0$

$$\Leftrightarrow \quad \boxed{\partial_\mu J^\mu = 0 \quad \text{mit} \quad \partial_\mu := \frac{\partial}{\partial x^\mu}.}$$

4-Potential

Definition : $A := \begin{pmatrix} \phi \\ \vec{A} \end{pmatrix}$.

Lorenz-Eichbedingung : $\frac{1}{c}\dot{\phi} + \nabla \cdot \vec{A} = 0$ (Seite 85)

$$\Leftrightarrow \quad \frac{\partial \phi}{\partial x^0} + \sum_i \frac{\partial A^i}{\partial x^i} = \partial_\mu A^\mu = 0 .$$

Maxwell-Gleichungen :

$$\begin{cases} \Box \phi = 4\pi \rho \\ \Box \vec{A} = \frac{4\pi}{c} \vec{J} \end{cases} \Rightarrow \boxed{\begin{array}{l} \Box A^\mu = \frac{4\pi}{c} J^\mu , \\ \Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \partial^\mu \partial_\mu \end{array}}$$

Bemerkung : $x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \Leftrightarrow \quad x^\nu = (\Lambda^{-1})^\nu{}_\mu x'^\mu$

$$\Rightarrow \partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \partial_\nu (\Lambda^{-1})^\nu{}_\mu$$

transformiert wie ein kovarianter Vektor !

$\Rightarrow \Box = \partial^\mu \partial_\mu$ ist 4-Skalar $\Rightarrow$ A^μ ist 4-Vektor wie J^μ .

Maxwell-Gleichungen mit 4-Potential aber ohne Eichbedingung:

Seite 85 :

$$\begin{cases} \Box \phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\dot{\phi} + \nabla \cdot \vec{A} \right) = 4\pi \rho \\ \Box \vec{A} + \nabla \left(\frac{1}{c}\dot{\phi} + \nabla \cdot \vec{A} \right) = \frac{4\pi}{c} \vec{J} \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_\mu \partial^\mu A^0 - \partial^0 \partial_\mu A^\mu = \frac{4\pi}{c} J^0 \\ \partial_\mu \partial^\mu A^i + \partial_i \partial_\mu A^\mu = \frac{4\pi}{c} J^i \end{cases} ; \quad \partial_i = \frac{\partial}{\partial x^i} = -\partial^i$$

$$\Leftrightarrow \quad \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = \frac{4\pi}{c} J^\nu , \quad \nu = 0,1,2,3$$

Bezeichne $F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$ „Feldstärketensor"

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$$

Hier ist :

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = \frac{1}{c}\dot{A}^i + \partial_i \phi \overset{!}{=} -E^i$$

$$F^{i0} = -F^{0i} = E^i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\left(\partial_i A^j - \partial_j A^i \right) \overset{!}{=} \begin{cases} -B^3 , & i=1, j=2 \\ -B^1 , & i=2, j=3 \\ -B^2 , & i=3, j=1 . \end{cases}$$

Maxwell-Gleichungen ohne Potentiale

Definiere einen Feldstärketensor mit
$$\begin{cases} F^{i0} := -F^{0i} := E^i \\ F^{ij} := -F^{ji} := -\varepsilon^{ijk} B^k \\ F^{00} := F^{ii} := 0 \end{cases}$$

Maxwell I $\quad \nabla \cdot \vec{E} = 4\pi \varrho \iff \sum_i \partial_i E^i = \frac{4\pi}{c} \varrho c \iff \partial_\mu F^{\mu 0} = \frac{4\pi}{c} J^0$

Maxwell II $\quad \nabla \times \vec{B} - \frac{1}{c}\dot{\vec{E}} = \frac{4\pi}{c}\vec{J} \iff \varepsilon^{jik}\partial_i B^k - \partial_0 E^j = \frac{4\pi}{c} J^j$

$$\iff \partial_i \underbrace{(-\varepsilon^{ijk}B^k)}_{F^{ij}} + \partial_0 \underbrace{(-E^j)}_{F^{0j}} = \frac{4\pi}{c} J^j \iff \partial_\mu F^{\mu j} = \frac{4\pi}{c} J^j$$

Maxwell I + II : $\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \quad , \nu = 0,\dots,3}$

Maxwell III $\quad \nabla \times \vec{E} + \frac{1}{c}\dot{\vec{B}} = 0 \quad \Big| \quad F^{ij} = -\varepsilon^{ijk} B^k \quad \Big| \cdot \varepsilon^{ijm}$

$$\varepsilon^{mij} F^{ij} = -2B^m$$

← Übungsblatt 10

$$\iff \underbrace{\varepsilon^{ijk}\partial_j E^k}_{\varepsilon^{ijk}\partial_j F^{k0}} - \frac{1}{2}\varepsilon^{ijk}\partial_0 F^{jk} = 0$$

$$\varepsilon^{ijk}\partial_j F^{k0} = -\frac{1}{2}\varepsilon^{ijk}\left(\partial^j F^{k0} + \partial^k F^{0j}\right)$$

$j \leftrightarrow k$

$$\iff -\frac{1}{2}\varepsilon^{ijk}\left(\partial^0 F^{jk} + \partial^j F^{k0} + \partial^k F^{0j}\right) = 0 \quad \Big| \ \varepsilon^{imn}$$

$$\iff \partial^0 F^{mn} + \partial^m F^{n0} + \partial^n F^{0m} = 0 \qquad \forall\ m,n$$

Maxwell IV $\quad \nabla \cdot \vec{B} = 0$

$$\iff \partial_m \varepsilon^{mij} F^{ij} = 0$$

$$\iff \varepsilon^{mij}\underbrace{\left(\partial^m F^{ij} + \partial^i F^{jm} + \partial^j F^{mi}\right)}_{} = 0$$

sowieso antisymmetrisch in allen Umtauschen,

z.B. $i \leftrightarrow j$: $\partial^m F^{ji} + \partial^j F^{im} + \partial^i F^{mj}$

$= -\partial^m F^{ij} - \partial^i F^{jm} - \partial^j F^{mi}$

$$\iff \partial^m F^{ij} + \partial^i F^{jm} + \partial^j F^{mi} = 0 \qquad \forall\ i,j,m$$

Maxwell III + IV : $\boxed{\partial^\mu F^{\nu\varrho} + \partial^\nu F^{\varrho\mu} + \partial^\varrho F^{\mu\nu} = 0 \quad \forall\ \mu,\nu,\varrho}$

Nachprüfung : $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \Rightarrow \cancel{\partial^\mu\partial^\nu A^\varrho} - \partial^\mu\partial^\varrho A^\nu + \partial^\nu\partial^\varrho A^\mu - \cancel{\partial^\nu\partial^\mu A^\varrho} + \partial^\varrho\partial^\mu A^\nu - \partial^\varrho\partial^\nu A^\mu = 0.$

Lorentz-Kraft

Bewegungsgleichung : $\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$; $d\tau = dt\sqrt{1-\frac{v^2}{c^2}}$ (Seite 104)

Hier ist $E^i = F^{i0}$ und

$$(\vec{v}\times\vec{B})^i = \varepsilon^{ijk}v^jB^k = -\frac{1}{2}\varepsilon^{ijk}\varepsilon^{klm}v^jF^{lm}$$
$$= -\frac{1}{2}\left(\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}\right)v^jF^{lm}$$
$$= -\frac{1}{2}F^{ij}v^j + \frac{1}{2}F^{ji}v^j = F^{ij}v_j$$

$$\Rightarrow \frac{dp^i}{d\tau} = \frac{q}{\sqrt{1-v^2/c^2}}\left(F^{i0} + \frac{1}{c}F^{ij}v_j\right) = \underline{\frac{q}{c}F^{i\nu}u_\nu}$$

Was passiert mit der 0-Komponente ?

Seite 109 : v^i

$$\frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau} = \frac{1}{c}\frac{\partial E}{\partial p^i}\frac{dp^i}{d\tau}$$
$$= \frac{1}{c}\cdot v^i\cdot\frac{q}{c}\cdot F^{i\nu}u_\nu$$
$$= \frac{q}{c^2}v^iF^{i0}u_0 - \frac{q}{c^2}\cdot\frac{1}{\sqrt{1-v^2/c^2}}\underbrace{v^iv^j}_{\text{symmetrisch}}\underbrace{F^{ij}}_{\text{antisymmetrisch}}$$
$$= \frac{q}{c}\cdot\frac{v^i}{\sqrt{1-\frac{v^2}{c^2}}}\cdot F^{i0}$$
$$= \frac{q}{c}\cdot u^iF^{i0} = \underline{\frac{q}{c}\cdot F^{0i}u_i}$$

Insgesamt :

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c}\cdot F^{\mu\nu}u_\nu}$$

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Fazit: Alles wird einfacher und schöner in der 4-Notation !

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